

Proofs, Arguments, and Zero-Knowledge¹

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Abstract

Interactive proofs (IPs) and arguments are cryptographic protocols that enable an untrusted prover to provide a guarantee that it performed a requested computation correctly. Introduced in the 1980s, IPs and arguments represented a major conceptual expansion of what constitutes a “proof” that a statement is true.

Traditionally, a proof is a static object that can be easily checked step-by-step for correctness. In contrast, IPs allow for interaction between prover and verifier, as well as a tiny but nonzero probability that an incorrect proof passes verification. Arguments (but not IPs) even permit there to be “proofs” of incorrect statements, so long as those “proofs” require exorbitant computational power to find. To an extent, these notions mimic in-person interactions that mathematicians use to convince each other that a claim is true, without going through the painstaking process of writing out and checking a traditional static proof.

Celebrated theoretical results from the 1980s and 1990s such as $\mathbf{IP} = \mathbf{PSPACE}$ and $\mathbf{MIP} = \mathbf{NEXP}$ showed that, in principle, surprisingly complicated statements can be verified efficiently. What is more, any argument can in principle be transformed into one that is *zero-knowledge*, which means that proofs reveal no information other than their own validity. Zero-knowledge arguments have a myriad of applications in cryptography.

Within the last decade, general-purpose zero-knowledge arguments have made the jump from theory to practice. This has opened new doors in the design of cryptographic systems, and generated additional insights into the power of IPs and arguments (zero-knowledge or otherwise). There are now no fewer than five promising approaches to designing efficient, general-purpose zero-knowledge arguments. This survey covers these approaches in a unified manner, emphasizing commonalities between them.

Preface and Acknowledgements

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A major benefit of taking 5 years (and counting) to complete this manuscript is the many exciting developments that can now be included. This survey would have looked very different had it been completed in 2015, or even in 2018 (perhaps 1/3 of the content covered did not exist 5 years ago). During this period, the various approaches to the design of zero-knowledge arguments, and the relationships between them, have come into finer focus. Yet owing to the sheer volume of research papers, it is increasingly challenging for those first entering the area to extract a clear picture from the literature itself.

Will the next 5-10 years bring a similar flood of developments? Will this be enough to render general-purpose arguments efficient enough for routine deployment in diverse cryptographic systems? It is my hope that this survey will make this exciting and beautiful area slightly more accessible, and thereby play some role in ensuring that the answer to both questions is "yes."

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Chapter 1

Introduction

This manuscript is about verifiable computing (VC). VC refers to cryptographic protocols called interactive proofs (IPs) and arguments that enable a prover to provide a guarantee to a verifier that the prover performed a requested computation correctly. Introduced in the 1980s, IPs and arguments represented a major conceptual expansion of what constitutes a “proof” that a statement is true. Traditionally, a proof is a static object that can be easily checked step-by-step for correctness, because each individual step of the proof should be trivial to verify. In contrast, IPs allow for interaction between prover and verifier, as well as a tiny but nonzero probability that an incorrect proof passes verification. The difference between IPs and arguments is that arguments (but not IPs) permit the existence of “proofs” of incorrect statements, so long as those “proofs” require exorbitant computational power to find.

Celebrated theoretical results from the mid-1980s and early 1990s indicated that VC protocols can, at least in principle, accomplish amazing feats. These include enabling a cell phone to monitor the execution of a powerful but untrusted (even malicious) supercomputer, enabling computationally weak peripheral devices (e.g., security card readers) to offload security-critical work to powerful remote servers, or letting a mathematician obtain a high degree of confidence that a theorem is true by looking at only a few symbols of a purported proof.¹

VC protocols can be especially useful in cryptographic contexts when they possess a property called *zero-knowledge*. This means that the proof or argument reveals nothing but its own validity.

To give a concrete sense of why zero-knowledge protocols are useful, consider the following quintessential example from authentication. Suppose that Alice chooses a random password x and publishes a hash $z = h(x)$, where h is a one-way function (this means that given $z = h(x)$ for a randomly chosen x , enormous computational power should be required to find a preimage of z under h , i.e., an x' such that $h(x') = z$). Later, suppose that Alice wants to convince Bob that she is the same person who published z . She can do this by proving to Bob that she knows an x' such that $h(x') = z$. This will convince Bob that Alice is the same person who published z , since it means that either Alice knew x to begin with, or she inverted h (which is assumed to be beyond the computational capabilities of Alice).

How can Alice convince Bob that she knows a preimage of z under h ? A trivial proof is for Alice to send x to Bob, and Bob can easily check that $h(x) = z$. But this reveals much more information than that Alice knows a preimage of z . In particular it reveals the preimage itself. Bob can use this knowledge to impersonate Alice forevermore, since now he too knows the preimage of z .

In order to prevent Bob from learning information that can compromise the password x , it is important

¹So long as the proof is written in a specific, mildly redundant format. See our treatment of *probabilistically checkable proofs* (PCPs) in Chapter 8.

that the proof reveals nothing beyond its own validity. This is exactly what the zero-knowledge property guarantees.

A particular goal of this survey is to describe a variety of approaches to constructing so-called zero-knowledge Succinct Non-interactive Arguments of Knowledge, or zk-SNARKs for short. “Succinct” means that the proofs are short. “Non-interactive” means that the proof is static, consisting of a single message from the prover. “Of Knowledge” roughly means that the protocol establishes not only that a statement is true, but also that the prover *knows* a “witness” to the veracity of the statement.² Argument systems satisfying all of these properties have a myriad of applications throughout cryptography.

Practical zero-knowledge protocols for highly specialized statements of cryptographic relevance (such as proving knowledge of a discrete logarithm [Sch89]) have been known for decades. However, general-purpose zero-knowledge protocols have only recently become plausibly efficient enough for cryptographic deployment. By general-purpose, we mean protocol design techniques that apply to arbitrary computations. This exciting progress has involved the introduction of beautiful new protocols, and brought a surge of interest in zero-knowledge proofs and arguments. This survey seeks to make accessible, in a unified manner, the main ideas and approaches to the design of these protocols.

Background and context. In the mid-1980s and 1990s, theoretical computer scientists showed that IPs and arguments can be vastly more efficient (at least, in an asymptotic sense) than traditional **NP** proofs, which are static and information-theoretically secure.³ The foundational results characterizing the power of these protocols (such as **IP=PSPACE** [LFKN92, Sha92], **MIP=NEXP** [BFL91], and the PCP theorem [ALM⁺98, AS98]) are some of the most influential and celebrated in computational complexity theory.⁴

Despite their remarkable asymptotic efficiency, general-purpose VC protocols were long considered wildly impractical, and with good reason: naive implementations of the theory would have had comically high concrete costs (trillions of years for the prover, even for very short computations). But the last decade has seen major improvements in the costs of VC protocols, with a corresponding jump from theory to practice. Even though implementations of general-purpose VC protocols remain somewhat costly (especially for the prover), paying this cost can often be justified if the VC protocol is zero-knowledge, since zero-knowledge protocols enable applications that may be totally impossible without them. Moreover, emerging applications to public blockchains have elevated the importance of proving relatively simple statements, on which it is feasible to run existing VC protocols despite their costs.

Approaches to zero-knowledge protocol design, and philosophy of this survey. Argument systems are typically developed in a two-step process. First, an information-theoretically secure protocol, such as an IP, *multi-prover interactive proof* (MIP), or *probabilistically checkable proof* (PCP), is developed for a model involving one or more provers that are assumed to behave in some restricted manner (e.g., in a MIP, the provers are assumed not to send information to each other about the challenges they receive from the verifier). Second, the information-theoretically secure protocol is combined with cryptography to “force” a (single) prover to behave in the restricted manner, thereby yielding an argument system. This second step also often endows the resulting argument system with important properties, such as zero-knowledge,

²For example, the authentication scenario above really requires a zero-knowledge proof *of knowledge* for the statement “there exists a password x such that $h(x) = z$ ”. This is because the application requires that Bob be convinced not just of the fact that there *exists* a preimage x of z under h (which will always be true if h is a surjective function), but also that Alice knows x .

³The term information-theoretically secure here refers to the fact that **NP** proofs (like IPs, but unlike arguments) are secure against computationally unbounded provers.

⁴The results **IP=PSPACE** and **MIP=NEXP** are both covered in this survey (see Sections 4.5.4 and 7.6 respectively).

succinctness, and non-interactivity. If the resulting argument satisfies all of these properties, then it is in fact a zk-SNARK.

By now, there are a variety promising approaches to developing efficient zk-SNARKs, which can be categorized by the type of information-theoretically secure protocol upon which they are based. These include (1) IPs, (2) MIPs, (3) PCPs, or more precisely a related notion called *interactive oracle proofs* (IOPs), which is a hybrid between an IP and a PCP, and (4) *linear PCPs*. Sections 1.2.1-1.2.3 below give a more detailed overview of these models.⁵ This survey explains in a unified manner how to design efficient protocols in all four information-theoretically secure models, emphasizing commonalities between them.

Ips, MIPs, and PCPs/IOPs can all be transformed into succinct interactive arguments by combining them with a cryptographic primitive called a *polynomial commitment scheme*; the interactive arguments can then be rendered non-interactive and publicly verifiable by applying a cryptographic technique called the *Fiat-Shamir transformation* (Section 4.7.2), yielding a SNARK. Transformations from linear PCPs to arguments are somewhat different: we will see that linear PCPs can be transformed into interactive arguments using *homomorphic encryption schemes*, or directly to publicly-verifiable SNARKs using *pairing-based cryptography*. As with the information-theoretically secure protocols themselves, this survey covers these cryptographic transformations in a unified manner.

Because of the two-step nature of zk-SNARK constructions, it is often helpful to first understand proofs and arguments *without* worrying about zero-knowledge, and then at the very end understand how to achieve zero-knowledge as an “add on” property. Accordingly, we do not discuss zero-knowledge until relatively late in this survey (Chapter 10). The bulk of the manuscript is devoted to describing efficient protocols in each of the information-theoretically secure models, and explaining how to transform each protocol into succinct arguments.

By now, zk-SNARKs have been deployed in a number of real-world systems, and there is a large and diverse community of researchers, industry professionals, and open source software developers working to improve and deploy the technology. This survey assumes very little mathematical background—mainly comfort with modular arithmetic and some notions from the theory of finite fields and groups—and is intended as a resource for anyone interested in verifiable computing and zero-knowledge. Also helpful is knowledge of standard complexity classes like **P** and **NP**, and complexity-theoretic notions such as **NP**-completeness. However, the main ideas covered in the survey can be understood even without prior knowledge of complexity theory. The primary prerequisite for the survey is significant mathematical maturity and considerable comfort with theorems and proofs.

Organization of this manuscript. The remainder of this introductory section provides more detail on the kinds of protocols studied in this survey. Chapter 2 familiarizes the reader with randomness and the power of probabilistic proof systems, through two easy but important case studies. Chapter 3 introduces technical notions that will be useful throughout the survey. Chapters 4 and 6 describe state-of-the-art interactive proofs and SNARKs derived thereof, while Chapter 5 describes techniques for representing computer programs in formats amenable to application of such arguments. Chapter 7 describes state-of-the-art MIPs and SNARKs derived thereof. Chapters 8-9 describe PCPs and IOPs, and SNARKs derived thereof. Chapter 10 introduces the notion of zero-knowledge. Chapter 11 describes a particularly simple type of zero-knowledge argument called Σ -protocols, and uses them to derive commitment schemes. These commitments scheme serve as important building blocks for the general-purpose zero-knowledge arguments constructed in later chapters. Chapter 12 describes efficient techniques for transforming non-zero-knowledge protocols into zero-knowledge ones. Chapter 13 covers practical polynomial commitment schemes, which can be used to

⁵There is not always a clean line to be drawn between the information-theoretically secure models. For example, both IPs and PCPs are special cases of IOPs. And even the MIPs that we cover can be transformed into IOPs (see Sections 9.5 and 9.6).

render any IP, MIP, or IOP into a succinct zero-knowledge argument of knowledge (zkSNARK). Chapter 14 covers our final approach to designing zkSNARKs, namely through linear PCPs. Chapter 15 describes how to recursively compose SNARKs to improve their costs and achieve important primitives such as so-called *incrementally verifiable computation*. Finally, Chapter 16 provides a taxonomy of design paradigms for practical zkSNARKs, and delineates the pros and cons of each approach.

Dependencies between chapters. Chapter 3 introduces basic technical notions used throughout all subsequent chapters (IPs, arguments, low-degree extensions, the Schwartz-Zippel lemma, etc.), and hence should not be skipped by readers unfamiliar with these notions. With these notions in hand, the subsequent chapters of this manuscript are mostly self-contained and should be accessible if read in isolation, with the following exceptions. Chapter 6 derives an argument system for circuit satisfiability based on the IPs of Chapter 4, and so should be read only after Chapter 4. Before reading Chapter 7 on MIPs, the reader should first tackle Section 4.1 on the sum-check protocol. Most of Chapter 9 on IOPs can be read in isolation, with the exception of Section 9.4; this presents an IOP for a problem called Reed-Solomon testing, for which it is beneficial to have first read Section 8.4.3 which covers a less efficient PCP for the same task. Readers encountering the notion of zero-knowledge for the first time must read Section 10.1 before tackling the zero-knowledge arguments given in Chapters 11 and 12, and in Section 14.5.5. The polynomial commitment schemes presented in Section 13.1 also cannot be understood without first reading Chapter 11. Finally, the linear-PCP-derived SNARK of Section 14.5 uses pairing-based cryptography that is introduced in Section 13.2.1.

1.1 Mathematical Proofs

This survey covers different notions of *mathematical proofs* and their applications in computer science and cryptography. Informally, what we mean by a proof is anything that convinces someone that a statement is true, and a “proof system” is any procedure that decides what is and is not a convincing proof. That is, a proof system is specified by a verification procedure that takes as input any statement and a claimed “proof” that the statement is true, and decides whether or not the proof is valid.

What properties do we want in a proof system? Here are four obvious ones.

- Any true statement should have a convincing proof of its validity. This property is typically referred to as *completeness*.
- No false statement should have a convincing proof. This property is referred to as *soundness*.
- Ideally, the verification procedure will be “efficient”. Roughly, this means that simple statements should have short (convincing) proofs that can be *checked* quickly.
- Ideally, proving should be efficient too. Roughly, this means that simple statements should have short (convincing) proofs that can be *found* quickly.

Traditionally, a mathematical proof is something that can be written and checked line-by-line for correctness. This traditional notion of proof is precisely the one captured by the complexity class **NP**. However, over the last 30+ years, computer scientists have studied much more general and exotic notions of proofs. This has transformed computer scientists’ notions of what it means to prove something, and has led to major advances in complexity theory and cryptography.

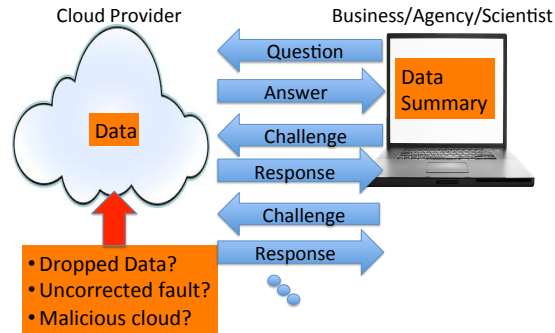


Figure 1.1: Depiction of an interactive proof or argument used to check that a cloud computing provider is storing and processing a user’s data correctly.

1.2 What kinds of non-traditional proofs will we study?

All of the notions of proofs that we study in this survey will be probabilistic in nature. This means that the verification procedure will make random choices, and the soundness guarantee will hold with (very) high probability over those random choices. That is, there will be a (very) small probability that the verification procedure will declare a false statement to be true.

1.2.1 Interactive Proofs (IPs)

To understand what an interactive proof is, it is helpful to think of the following application. Imagine a business (verifier) that is using a commercial cloud computing provider to store and process its data. The business sends all of its data up to the cloud (prover), which stores it, while the business stores only a very small “secret” summary of the data (meaning that the cloud does not know the user’s secret summary). Later, the business asks the cloud a question about its data, typically in the form of a computer program f that the business wants the cloud to run on its data using the cloud’s vast computing infrastructure. The cloud does so, and sends the user the claimed output of the program, $f(\text{data})$. Rather than blindly trust that the cloud executed the program on the data correctly, the business can use an interactive proof system (IP) to obtain a formal *guarantee* that the claimed output is correct.

In the IP, the business interrogates the cloud, sending a sequence of challenges and receiving a sequence of responses. At the end of the interrogation, the business must decide whether to accept the answer as valid or reject it as invalid. See Figure 1.1 for a diagram of this interaction.

Completeness of the IP means that if the cloud correctly runs the program on the data and follows the prescribed protocol, then the user will be convinced to accept the answer as valid. Soundness of the IP means that if the cloud returns the wrong output, then the user will reject the answer as invalid with high probability *no matter how hard the cloud works to trick the user* into accepting the answer as valid. Intuitively, the interactive nature of the IP lets the business exploit the element of surprise (i.e., the fact that the cloud cannot predict the business’s next challenge) to catch a lying cloud in a lie.

It is worth remarking on an interesting difference between IPs and traditional static proofs. Static proofs are *transferrable*, meaning that if Peggy (prover) hands Victor (verifier) a proof that a statement is true, Victor can turn around and convince Tammy (a third party) that the same statement is true, simply by

copying the proof. In contrast, an interactive proof may not be transferrable. Victor can try to convince Tammy that the statement is true by sending Tammy a transcript of his interaction with Peggy, but Tammy will not be convinced unless Tammy trusts that Victor correctly represented the interaction. This is because soundness of the IP only holds if, every time Peggy sends a response to Victor, Peggy does not know what challenge Victor will respond with next. The transcript alone does not give Tammy a guarantee that this holds.

1.2.2 Argument Systems

Argument systems are IPs, but where the soundness guarantee need only hold against cheating provers that run in polynomial time.⁶ Argument systems make use of cryptography. Roughly speaking, in an argument system a cheating prover cannot trick the verifier into accepting a false statement unless it breaks some cryptosystem, and breaking the cryptosystem is assumed to require superpolynomial time.

1.2.3 Multi-Prover Interactive Proofs, Probabilistically Checkable Proofs, etc.

A MIP is like an IP, except that there are multiple provers, and these provers are assumed not to share information with each other regarding what challenges they receive from the verifier. A common analogy for MIPs is placing two or more criminal suspects in separate rooms before interrogating them, to see if they can keep their story straight. Law enforcement officers may be unsurprised to learn that the study of MIPs has lent theoretical justification to this practice. Specifically, the study of MIPs has revealed that if one locks the provers in separate rooms and then interrogates them separately, they can convince their interrogators of much more complicated statements than if they are questioned together.

In a PCP, the proof is static as in a traditional mathematical proof, but the verifier is only allowed to read a small number of (possibly randomly chosen) characters from the proof. This is in analogy to a lazy referee for a mathematical journal, who does not feel like painstakingly checking the proofs in a submitted paper for correctness. The PCP theorem [ALM⁺98, AS98] essentially states that *any* traditional mathematical proof can be written in a format that enables this lazy reviewer to obtain a high degree of confidence in the validity of the proof by inspecting just a few words of it.

Philosophically, MIPs and PCPs are extremely interesting objects to study, but they are not directly applicable in most cryptographic settings, because they make unrealistic or onerous assumptions about the prover(s). For example, soundness of any MIP only holds if the provers do not share information with each other regarding what challenges they receive from the verifier. This is not directly useful in most cryptographic settings, because typically in these settings there is only a single prover, and even if there is more than one, there is no way to force the provers not to communicate. Similarly, although the verifier only reads a few characters of a PCP, a direct implementation of a PCP would require the prover to transmit the whole proof to the verifier, and this would be the dominant cost in most real-world scenarios (the example of a lazy journal referee notwithstanding). That is, once the prover transmits the whole proof to the verifier, there is little real-world benefit to having the verifier avoid reading the whole proof.

However, by combining MIPs and PCPs with cryptography, we will see how to turn them into argument systems, and these *are* directly applicable in cryptographic settings. For example, we will see in Section 8.2 how to turn a PCP into an argument system in which the prover does *not* have to send the whole PCP to the verifier.

⁶Roughly speaking, this means that if the input has size n , then the prover's runtime (for sufficiently large values of n) should be bounded above by some constant power of n , e.g., n^{10} .

Chapter 2

The Power of Randomness: Fingerprinting and Freivalds' Algorithm

2.1 Reed-Solomon Fingerprinting

The proof systems covered in this survey derive much of their power and efficiency from their use of randomness. Before we discuss the details of such proof systems, let us first develop an appreciation for how randomness can be exploited to dramatically improve the efficiency of certain algorithms. Accordingly, in this section, there are no untrusted provers or computationally weak verifiers. Rather, we consider two parties, Alice and Bob, who trust each other and want to cooperate to jointly compute a certain function of their inputs.

2.1.1 The Setting

Alice and Bob live across the country from each other. They each hold a very large file, each consisting of n characters (for concreteness, suppose that these are ASCII characters, so there are $m = 128$ possible characters). Let us denote Alice's file as the sequence of characters (a_1, \dots, a_n) , and Bob's as (b_1, \dots, b_n) . Their goal is to determine whether their files are *equal*, i.e., whether $a_i = b_i$ for all $i = 1, \dots, n$. Since the files are large, they would like to minimize *communication*, i.e., Alice would like to send as little information about her file to Bob as possible.

A trivial solution to this problem is for Alice to send her entire file to Bob, and Bob can check whether $a_i = b_i$ for all $i = 1, \dots, n$. But this requires Alice to send all n characters to Bob, which is prohibitive if n is very large. It turns out that no *deterministic* procedure can send less information than this trivial solution.⁷

However, we will see that if Alice and Bob are allowed to execute a *randomized* procedure that might output the wrong answer with some tiny probability, say at most 0.0001, then they can get away with a much smaller amount of communication.

2.1.2 The Communication Protocol

The High-Level Idea. The rough idea is that Alice is going to pick a hash function h at random from a (small) family of hash functions \mathcal{H} . We will think of $h(x)$ as a very short “fingerprint” of x . By fingerprint,

⁷The interested reader is directed to [KN97, Example 1.2.1] for a proof of this fact, based on the so-called *fooling set method* in communication complexity.

we mean that $h(x)$ is a “nearly unique identifier” for x , in the sense that for any $y \neq x$, the fingerprints of x and y differ with high probability over the random choice of h , i.e.,

$$\text{for all } x \neq y, \Pr_{h \in \mathcal{H}} [h(x) = h(y)] \leq 0.0001.$$

Rather than sending a to Bob in full, Alice sends h and $h(a)$ to Bob. Bob checks whether $h(a) = h(b)$. If $h(a) \neq h(b)$, then Bob *knows* that $a \neq b$, while if $h(a) = h(b)$, then Bob can be very confident (but not 100% sure) that $a = b$.

The Details. To make the above outline concrete, fix a prime number $p > \max\{m, n^2\}$, and let \mathbb{F}_p denote the set of integers modulo p . For the remainder of this section, we assume that all arithmetic is done *modulo* p without further mention.⁸ This means that all numbers are replaced with their remainder when divided by p . So, for example, if $p = 17$, then $(2 \cdot 3^2 + 4) \pmod{17} = 22 \pmod{17} = 5$.

For each $r \in \mathbb{F}_p$, define $h_r(a_1, \dots, a_n) = \sum_{i=1}^n a_i \cdot r^i$. The family \mathcal{H} of hash functions we will consider is

$$\mathcal{H} = \{h_r : r \in \mathbb{F}_p\}. \tag{2.1}$$

Intuitively, each hash function h_r interprets its input (a_1, \dots, a_n) as the coefficients of a degree n polynomial, and outputs the polynomial evaluated at r .

That is, in our communication protocol, Alice picks a random element r from \mathbb{F}_p , computes $v = h_r(a)$, and sends v and r to Bob. Bob outputs EQUAL if $v = h_r(b)$, and outputs NOT-EQUAL otherwise.

2.1.3 The Analysis

We now prove that this protocol outputs the correct answer with very high probability. In particular:

- If $a_i = b_i$ for all $i = 1, \dots, n$, then Bob outputs EQUAL for every possible choice of r .
- If there is even one i such that $a_i \neq b_i$, then Bob outputs NOT-EQUAL with probability at least $1 - n/p$, which is at least $1 - 1/n$ by choice of $p > n^2$.

The first property is easy to see: if $a = b$, then obviously $h_r(a) = h_r(b)$ for every possible choice of r . The second property relies on the following crucial fact, whose validity we justify later in Section 2.1.6.

Fact 2.1. *For any two distinct (i.e., unequal) polynomials p_a, p_b of degree at most n with coefficients in \mathbb{F}_p , $p_a(x) = p_b(x)$ for at most n values of x in \mathbb{F}_p .*

Let $p_a(x) = \sum_i a_i \cdot x^i$ and similarly $p_b(x) = \sum_i b_i \cdot x^i$. Observe that both p_a and p_b are polynomials in x of degree at most n . The value v that Alice sends to Bob in the communication protocol is precisely $p_a(r)$, and Bob compares this value to $p_b(r)$.

By Fact 2.1, if there is even one i such that $a_i \neq b_i$, then there are at most n values of r such that $p_a(r) = p_b(r)$. Since r is chosen at random from \mathbb{F}_p , the probability that Alice picks such an r is thus at most n/p . Hence, Bob outputs NOT-EQUAL with probability at least $1 - n/p$ (where the probability is over the random choice of r).

⁸The reason to perform all arithmetic modulo p rather than over the integers is to ensure that all numbers arising in the protocol can always be represented using just $\log_2(p) = O(\log(n) + \log(m))$ bits. If arithmetic was performed over the integers rather than modulo p , then the protocol covered in this section would require Alice to send to Bob an integer that may have magnitude more than 2^n , which would require more than n bits to represent. This is nearly as expensive as having Alice send her entire input to Bob.

2.1.4 Cost of the Protocol

Alice sends only two elements of \mathbb{F}_p to Bob in the above protocol, namely v and r . In terms of bits, this is $O(\log n)$ bits assuming $p \leq n^c$ for some constant c . This is an *exponential improvement* over the $n \cdot \log m$ bits sent in the deterministic protocol (all logarithms in this manuscript are to base 2 unless the base is explicitly specified otherwise). This is an impressive demonstration of the power of randomness.⁹

2.1.5 Discussion

We refer to the above protocol as Reed-Solomon fingerprinting because $p_a(r)$ is actually a random entry in an *error-corrected encoding* of the vector (a_1, \dots, a_n) . The encoding is called the Reed-Solomon encoding. Several other fingerprinting methods are known. Indeed, all that we really require of the hash family \mathcal{H} used in the protocol above is that for any $x \neq y$, $\Pr_{h \in \mathcal{H}}[h(x) = h(y)]$ is small. Many hash families are known to satisfy this property,¹⁰ but Reed-Solomon fingerprinting will prove particularly relevant in our study of probabilistic proof systems, owing to its algebraic structure.

A few sentences on finite fields. For prime p , \mathbb{F}_p is an example of a *field*, which is any set equipped with addition, subtraction, multiplication, and division operations, and such that these operations behave roughly the same as they do over the rational numbers.¹¹ So, for example, the set of real numbers is a field, because for any two real numbers c and d , it holds that $c + d$, $c - d$, $c \cdot d$, and (assuming $d \neq 0$) c/d are themselves all real numbers. The same holds for the set of complex numbers, and the set of rational numbers. In contrast, the set of integers is *not* a field, since dividing two integers does not necessarily yield another integer.

\mathbb{F}_p is also a field (a *finite* one). Here, the field operations are simply addition, subtraction, multiplication, and division modulo p . What we mean by division modulo p requires some explanation: for every $a \in \mathbb{F}_p$, there is a unique element $a^{-1} \in \mathbb{F}_p$ such that $a \cdot a^{-1} = 1$. For example, if $p = 5$ and $a = 3$, then $a^{-1} = 2$, since $3 \cdot 2 \pmod{5} = 6 \pmod{5} = 1$. Division by a in \mathbb{F}_p refers to multiplication by a^{-1} . So if $p = 5$, then in \mathbb{F}_p , $4/3 = 4 \cdot 3^{-1} = 4 \cdot 2 = 3$.

Much later in this manuscript (e.g., Section 13.2.1), we will exploit the fact that for any prime *power* (i.e., p^k for some prime p and positive integer k), there is a unique finite field of size p^k , denoted \mathbb{F}_{p^k} .¹²

⁹Readers familiar with cryptographic hash functions such as SHA-3 may be in the habit of thinking of such a hash function as a fixed, deterministic function, and hence perplexed by the characterization of our protocol as randomized (as Alice just sends the hash function h and the evaluation $h(a)$ to Bob, where a is Alice's input vector). To this, we offer two clarifications. First, the communication protocol in this section actually does not require a cryptographic hash function, but rather uses a function chosen at random from the hash family given in Equation (2.1), which is in fact far simpler than any cryptographic hash family, e.g., it is not collision-resistant or one-way. Second, cryptographic hash functions such as SHA-3 really should be modeled as having been sampled at random from some large family. Otherwise, properties such as collision-resistance would be broken against non-uniform adversaries (i.e., adversaries permitted unlimited pre-processing). For example, collision-resistance of any fixed deterministic function h is broken by simply "hard-coding" into the adversary two inputs x, x' such that $h(x) \neq h(x')$. This pre-processing attack does not work if h is chosen at random from a large family of functions, and the pre-processing has to occur prior to the random selection of h .

¹⁰Such hash families are called *universal*. The excellent Wikipedia article on universal hashing contains many constructions https://en.wikipedia.org/wiki/Universal_hashing.

¹¹In more detail, the addition and multiplication operations in any field must be associative and commutative. They must also satisfy the distributive law, i.e., $a \cdot (b + c) = a \cdot b + a \cdot c$. Moreover, there must be two special elements in the field, denoted 0 and 1, that are additive and multiplicative identity elements, i.e., for all field elements a , it must hold that $a + 0 = a$ and $a \cdot 1 = a$. Every field element a must have an additive inverse, i.e., a field element $-a$ such that $a + (-a) = 0$ (this ensures that subtraction can be defined in terms of addition of an additive inverse, i.e., $b - a$ is defined as $b + (-a)$). And every *nonzero* field element a must have a multiplicative inverse a^{-1} such that $a \cdot a^{-1} = 1$ (this ensures that division by a nonzero field element a can be defined as multiplication by a^{-1}).

¹²More precisely, all finite fields of size p^k are isomorphic, roughly meaning they have the exact same structure, though they

2.1.6 Establishing Fact 2.1

Fact 2.1 is implied by (in fact, equivalent to) the following fact.

Fact 2.2. *Any nonzero polynomial of degree at most n over any field has at most n roots.*

A simple proof of Fact 2.2 can be found online at [hp]. To see that Fact 2.2 implies Fact 2.1, observe that if p_a and p_b are *distinct* polynomials of degree at most n , and $p_a(x) = p_b(x)$ for more than n values of $x \in \mathbb{F}_p$, then $p_a - p_b$ is a *nonzero* polynomial of degree at most n with more than n roots.

2.2 Freivalds' Algorithm

In this section, we see our first example of an efficient probabilistic proof system.

2.2.1 The Setting

Suppose we are given as input two $n \times n$ matrices A and B over \mathbb{F}_p , where $p > n^2$ is a prime number. Our goal is to compute the product matrix $A \cdot B$. Asymptotically, the fastest known algorithm for accomplishing this task is very complicated, and runs in time roughly $O(n^{2.37286})$ [LG14, AW21]. Moreover, the algorithm is not practical. But for the purposes of this manuscript, the relevant question is not how fast can we multiply two matrices—it's how efficiently can one *verify* that two matrices were multiplied correctly. In particular, can verifying the output of a matrix multiplication problem be done faster than the fastest known algorithm for actually multiplying the matrices? The answer, given by Freivalds in 1977 [Fre77], is yes.

Formally, suppose someone hands us a matrix C , and we want to check whether or not $C = A \cdot B$. Here is a very simple randomized algorithm that will let us perform this check in $O(n^2)$ time¹³ (this is only a constant factor more time than what is required to simply read the matrices A, B , and C).

2.2.2 The Algorithm

First, choose a random $r \in \mathbb{F}_p$, and let $x = (r, r^2, \dots, r^n)$. Then compute $y = Cx$ and $z = A \cdot Bx$, outputting YES if $y = z$ and NO otherwise.

2.2.3 Runtime

We claim that the entire algorithm runs in time $O(n^2)$. It is easy to see that generating the vector $x = (r, r^2, \dots, r^n)$ can be done with $O(n)$ total multiplication operations (r^2 can be computed as $r \cdot r$, then r^3 can be computed as $r \cdot r^2$, then r^4 as $r \cdot r^3$, and so on). Since multiplying an $n \times n$ matrix by an n -dimensional vector can be done in $O(n^2)$ time, the remainder of the algorithm runs in $O(n^2)$ time: computing y involves multiplying C by the vector x , and computing $A \cdot Bx$ involves multiplying B by x to get a vector $w = Bx$, and then multiplying A by w to compute $A \cdot Bx$.

2.2.4 Completeness and Soundness Analysis

Let $D = A \cdot B$, so that our goal is to determine whether the *claimed* product matrix C actually equals the *true* product matrix D . Letting $[n]$ denote the set $\{1, 2, \dots, n\}$, we claim that the above algorithm satisfies the following two conditions:

may not assign names to elements in the same manner.

¹³Throughout this manuscript, we assume that addition and multiplication operations in finite fields take constant time.

- If $C = D$, then the algorithm outputs YES for every possible choice of r .
- If there is even one $(i, j) \in [n] \times [n]$ such that $C_{i,j} \neq D_{i,j}$, then Bob outputs NO with probability at least $1 - n/p$.

The first property is easy to see: if $C = D$, then clearly $Cx = Dx$ for all vectors x , so the algorithm will output YES for every choice of r . To see that the second property holds, suppose that $C \neq D$, and let C_i and D_i denote the i th row of C and D respectively. Obviously, since $C \neq D$, there is some row i such that $C_i \neq D_i$. Recalling that $x = (r, r^2, \dots, r^n)$, observe that $(Cx)_i$ is precisely $p_{C_i}(r)$, the Reed-Solomon fingerprint of C_i as in the previous section. Similarly, $(A \cdot B \cdot x)_i = p_{D_i}(r)$. Hence, by the analysis of Section 2.1.3, the probability that $(Cx)_i \neq (A \cdot B \cdot x)_i$ is at least $1 - n/p$, and in this event the algorithm outputs NO.

2.2.5 Discussion

Whereas fingerprinting saved communication compared to a deterministic protocol, Freivalds' algorithm saves *runtime* compared to the best known deterministic algorithm. We can think of Freivalds' algorithm as our first probabilistic proof system: here, the proof is simply the answer C itself, and the $O(n^2)$ -time verification procedure simply checks whether $Cx = A \cdot Bx$.

Freivalds actually described his algorithm with a perfectly random vector $x \in \mathbb{F}_p^n$, rather than $x = (r, r^2, \dots, r^n)$ for a random $r \in \mathbb{F}_p$ (see Exercise 3.1). We chose $x = (r, r^2, \dots, r^n)$ to ensure that $(Cx)_i$ is a Reed-Solomon fingerprint of row i of C , thereby allowing us to invoke the analysis from Section 2.1.

Chapter 3

Definitions and Technical Preliminaries

3.1 Interactive Proofs

Given a function f mapping $\{0, 1\}^n$ to a finite range \mathcal{R} , a k -message *interactive proof system* (IP) for f consists of a probabilistic verifier algorithm \mathcal{V} running in time $\text{poly}(n)$ and a prescribed (“honest”) deterministic prover algorithm \mathcal{P} .^{14,15} Both \mathcal{V} and \mathcal{P} are given a common input $x \in \{0, 1\}^n$, and at the start of the protocol \mathcal{P} provides a value y claimed to equal $f(x)$. Then \mathcal{P} and \mathcal{V} exchange a sequence of messages m_1, m_2, \dots, m_k that are determined as follows. The IP designates one of the parties, either \mathcal{P} or \mathcal{V} , to send the first message m_1 . The party sending each message alternates, meaning for example that if \mathcal{V} sends m_1 , then \mathcal{P} sends m_2 , \mathcal{V} sends m_3 , \mathcal{P} sends m_4 , and so on.¹⁶

Both \mathcal{P} and \mathcal{V} are thought of as “next-message-computing algorithms”, meaning that when it is \mathcal{V} ’s (respectively, \mathcal{P} ’s) turn to send a message m_i , \mathcal{V} (respectively, \mathcal{P}) is run on input $(x, m_1, m_2, \dots, m_{i-1})$ to produce message m_i . Note that since \mathcal{V} is probabilistic, any message m_i sent by \mathcal{V} may depend on both $(x, m_1, m_2, \dots, m_{i-1})$ and on the verifier’s internal randomness.

The entire sequence of k messages $t := (m_1, m_2, \dots, m_k)$ exchanged by \mathcal{P} and \mathcal{V} , along with the claimed answer y , is called a *transcript*. At the end of the protocol, \mathcal{V} must output either 0 or 1, with 1 indicating that the verifier accepts the prover’s claim that $y = f(x)$ and 0 indicating that the verifier rejects the claim. The value output by the verifier at the end of the protocol may depend on both the transcript t and the verifier’s internal randomness.

Denote by $\text{out}(\mathcal{V}, x, r, \mathcal{P}) \in \{0, 1\}$ the output of verifier \mathcal{V} on input x when interacting with deterministic prover strategy \mathcal{P} , with \mathcal{V} ’s internal randomness equal to r . For any fixed value r of \mathcal{V} ’s internal randomness, $\text{out}(\mathcal{V}, x, r, \mathcal{P})$ is a deterministic function of x (as we have restricted our attention to deterministic prover strategies \mathcal{P}).

Definition 3.1. An interactive proof system $(\mathcal{V}, \mathcal{P})$ is said to have completeness error δ_c and soundness error δ_s if the following two properties hold.

¹⁴In general, one may consider defining IPs to permit probabilistic prover strategies. However, as explained in Section 3.3, it is without loss of generality to restrict attention to deterministic prover strategies.

¹⁵The choice of domain $\{0, 1\}^n$ in this chapter is not essential, but rather made by convention and for convenience. One reason $\{0, 1\}^n$ is a convenient domain is that, in order to express a proof system’s costs (e.g., prover time and verifier time) in terms of the size of the input, we need a well-defined notion of input size, and if the input domain is all n -bit strings, then n is the natural such measure.

¹⁶Without loss of generality, the final message m_k is sent by the prover. There is no point in having the verifier send a message to the prover if the prover is not going to respond to it.

1. (*Completeness*) For every $x \in \{0, 1\}^n$,

$$\Pr_r[\text{out}(\mathcal{V}, x, r, \mathcal{P}) = 1] \geq 1 - \delta_c.$$

2. (*Soundness*) For every $x \in \{0, 1\}^n$ and every deterministic prover strategy \mathcal{P}' , if \mathcal{P}' sends a value $y \neq f(x)$ at the start of the protocol, then

$$\Pr_r[\text{out}(\mathcal{V}, x, r, \mathcal{P}') = 1] \leq \delta_s.$$

An interactive proof system is valid if $\delta_c, \delta_s \leq 1/3$.

Intuitively, for any input x , the completeness condition requires that there be a convincing proof for what is the value of f on input x . The soundness condition requires that false statements of the form “ $f(x) = y$ ” for any $y \neq f(x)$ lack a convincing proof. That is, there is no cheating prover strategy \mathcal{P}' that can convince \mathcal{V} to accept a false claim with probability more than $1/3$.

The two costs of paramount importance in any interactive proof are \mathcal{P} ’s runtime and \mathcal{V} ’s runtime, but there are other important costs as well: \mathcal{P} ’s and \mathcal{V} ’s space usage, the total number of bits communicated, and the total number of messages exchanged. If \mathcal{V} and \mathcal{P} exchange k messages, then $\lceil k/2 \rceil$ is referred to as the *round complexity* of the interactive proof system.¹⁷ The round complexity is the number of “back-and-forths” in the interaction between \mathcal{P} and \mathcal{V} . If k is odd, then the final “back-and-forth” in the interaction is really just a “back” with no “forth”, i.e., it consists of only one message from prover to verifier.

Interactive proofs were introduced in 1985 by Goldwasser, Micali, and Rackoff [GMR89] and Babai [Bab85].¹⁸

3.2 Argument Systems

Definition 3.2. An *argument system* for a function f is an interactive proof for f in which the soundness condition is only required to hold against prover strategies that run in polynomial time.

The notion of soundness in Definition 3.2 is called *computational soundness*. Computational soundness should be contrasted with the notion of soundness in Definition 3.1, which is required to hold even against computationally unbounded provers \mathcal{P}' that might be devoting enormous computational resources to trying to trick \mathcal{V} into accepting an incorrect answer. The soundness notion from Definition 3.1 is referred to as *statistical soundness* or *information-theoretic soundness*.

Argument systems were introduced by Brassard, Chaum, and Crépeau in 1986 [BCC88]. They are sometimes referred to as *computationally sound proofs*, but in this manuscript we will mainly use the term “proof” to refer to statistically sound protocols.¹⁹ Unlike interactive proofs, argument systems are able to utilize cryptographic primitives. While a super-polynomial time prover may be able to break the primitive

¹⁷Be warned that the literature is not consistent with regard to the meaning of the term “rounds”. Vexingly, many papers use the terms rounds and messages interchangeably.

¹⁸More precisely, [GMR89] introduced IPs, while Babai (with different motivations) introduced the so-called *Arthur-Merlin class hierarchy*, which captures constant-round interactive proof systems, with the additional requirement that the verifier’s randomness is public—that is, any coin tossed by \mathcal{V} is made visible to the prover as soon as it is tossed. See Section 3.3 for discussion of public vs. private verifier randomness.

¹⁹The main exception is Chapter 15, where we use the term “SNARK proof π ” to refer to a string π that convinces the verifier of a non-interactive argument system to accept. This terminology is unambiguous because the acronym SNARK, which is short for Succinct Non-interactive ARGument of Knowledge, clarifies that protocol at hand is an argument system.

and thereby trick the verifier into accepting an incorrect answer, a polynomial time prover will be unable to break the primitive. The use of cryptography often allows argument systems to achieve additional desirable properties that are unattainable for interactive proofs, such as reusability (i.e., the ability for the verifier to reuse the same “secret state” to outsource many computations on the same input), public verifiability, etc. These properties will be discussed in more detail later in this survey.

3.3 Robustness of Definitions and the Power of Interaction

At first glance, it may seem that a number of aspects of Definitions 3.1 and 3.2 are somewhat arbitrary or unmotivated. For example, why does Definition 3.1 insist that the soundness and completeness errors be at most $1/3$, and not some smaller number? Why does the completeness condition in Definition 3.1 demand that the honest prover is deterministic? And so forth. As we explain in this section, many of these choices are made for convenience or aesthetic reasons—the power of IPs and arguments are largely unchanged if different choices are made in the definitions.²⁰ The remarks in this section are somewhat technical and may be skipped with no loss of continuity.

- (Perfect vs. Imperfect Completeness) While Definition 3.1 required that the completeness error $\delta_c < 1/3$, all of the interactive proofs that we will see in this manuscript actually satisfy *perfect* completeness, meaning that $\delta_c = 0$. That is, the honest prover in our IPs and arguments will *always* convince the verifier that it is honest.

It is actually known [FGM⁺89] that any IP for a function f with $\delta_c \leq 1/3$ can be transformed into an IP for f with perfect completeness, with a polynomial blowup in the verifier’s costs (e.g., verifier time, round complexity, communication complexity).²¹ We will not need such transformations in this manuscript, because the IPs we give will naturally satisfy perfect completeness.

- (Soundness Error) While Definition 3.1 required the soundness error δ_s to be at most $1/3$, the constant $1/3$ is merely chosen by convention. In all of the interactive proofs that we see in this survey, the soundness error will always be proportional to $1/|\mathbb{F}|$, where \mathbb{F} is the field over which the interactive proof is defined. In practice, the field will typically be chosen large enough so that the soundness error is astronomically small (e.g., smaller than, say, 2^{-128}). Such tiny soundness error is essential in cryptographic applications, where a cheating prover successfully tricking a verifier to accept a false claim can have catastrophic effects. Soundness error of any IP or argument can also be generically reduced from δ_s to δ_s^k by repeating the protocol $\Theta(k)$ times in sequence and rejecting unless the verifier accepts in a majority of the repetitions.²²
- (Public vs. Private Randomness) In an interactive proof system, \mathcal{V} ’s randomness is internal, and in particular is not visible to the prover. This is referred to in the literature as *private randomness*. One can also consider IPs in which the verifier’s randomness is public—that is, any coin tossed by \mathcal{V} is made visible to the prover as soon as it is tossed. We will see that such *public-coin* IPs are particularly

²⁰Generally speaking, robustness to tweaks in the definition is a hallmark of a “good” notion or model in complexity theory. If the power of a model is highly sensitive to idiosyncratic or arbitrary choices in its definition, then the model may have limited utility and be unlikely to capture fundamental real-world phenomena. After all, the real world is messy and evolving—the hardware people use to compute is complicated and changes over time, protocols get used in a variety of different settings, etc. Robustness of a model to various tweaks helps ensure that any protocols in the model are useful in a variety of different settings and will not be rendered obsolete by future changes in technology.

²¹The transformation does *not* necessarily preserve the prover’s runtime.

²²For perfectly complete protocols, the verifier may reject unless *every* repetition of the base protocol leads to acceptance.

useful, because they can be combined with cryptography to obtain argument systems with important properties (see Sections 4.7.2 and 11.2.3 on the Fiat-Shamir transformation).

Goldwasser and Sipser [GS86] showed that the distinction between public and private coins is not crucial: any private coin interactive proof system can be simulated by a public coin system (with a polynomial blowup in costs for the verifier, and a small increase in the number of rounds). As with perfect vs. imperfect completeness, we will not need to utilize such transformations in this manuscript because all of the IPs that we give are naturally public coin protocols.

- (Deterministic vs. Probabilistic Provers) Definition 3.1 demands that the honest prover strategy \mathcal{P} be deterministic, and only requires soundness to hold against deterministic cheating prover strategies \mathcal{P}' . Restricting attention to deterministic prover strategies in this manner is done only for convenience, and does not alter the power of interactive proofs.

Specifically, if there is a probabilistic prover strategy \mathcal{P}' that convinces the verifier \mathcal{V} to accept with probability at least p (with the probability taken over both the prover’s internal randomness and the verifier’s internal randomness), then there is a deterministic prover strategy achieving the same. This follows from an averaging argument over the prover’s randomness: if a probabilistic prover \mathcal{P}' convinces \mathcal{V} to accept a claim “ $f(x) = y$ ” with probability p , there must be at least one setting of the internal randomness r' of \mathcal{P}' such that the deterministic prover strategy obtained by fixing the randomness of \mathcal{P}' to r' also convinces the verifier to accept the claim “ $f(x) = y$ ” with probability p . (Note that the value r' may depend on x). In this manuscript, the honest prover in all of our IPs and arguments will naturally be deterministic, so we will have no need to exploit this generic transformation from randomized to deterministic prover strategies.²³

Interactive Proofs for Languages Versus Functions. Complexity theorists often find it convenient to study *decision problems*, which are functions f with range $\{0, 1\}$. We think of decision problems as “yes-no questions”, in the following manner: any input x to f is interpreted as a question, namely: “Does $f(x)$ equal 1?”. Equivalently, we can associate any decision problem f with the subset $\mathcal{L} \subseteq \{0, 1\}^n$ consisting of “yes-instances” for f . Any subset $\mathcal{L} \subseteq \{0, 1\}^n$ is called a *language*.

The formalization of IPs for languages differs slightly from that for functions (Definition 3.1). We briefly describe this difference because celebrated results in complexity theory regarding the power of IPs and their variants (e.g., $\mathbf{IP} = \mathbf{PSPACE}$ and $\mathbf{MIP} = \mathbf{NEXP}$) refer to IPs for languages.

In an interactive proof for the language \mathcal{L} , given a public input $x \in \{0, 1\}^n$, the verifier \mathcal{V} interacts with a prover \mathcal{P} in exactly the same manner as in Definition 3.1 and at the end of the protocol \mathcal{V} must output either 0 or 1, with 1 corresponding to “accept” and 0 corresponding to “reject”. The standard requirements of an IP for the language \mathcal{L} are:

- **Completeness.** For any $x \in \mathcal{L}$, there is some prover strategy that will cause the verifier to accept with high probability.
- **Soundness.** For any $x \notin \mathcal{L}$, then for *every* prover strategy, the verifier rejects with high probability.

Given a language \mathcal{L} , let $f_{\mathcal{L}}: \{0, 1\}^n \rightarrow \{0, 1\}$ be the corresponding decision problem, i.e., $f_{\mathcal{L}}(x) = 1$ if x is in \mathcal{L} , and $f_{\mathcal{L}}(x) = 0$ if x is not in \mathcal{L} . Note that for $x \notin \mathcal{L}$, the above definition of an IP for \mathcal{L} does *not*

²³An important caveat is that for most of the *zero-knowledge* proofs and arguments considered in Chapters 10-14 in this manuscript, the prover will be randomized. This randomization of the proof has no bearing on the completeness or soundness of the protocol, but rather is incorporated as a means of ensuring that the proof leaks no information to the verifier (other than its own validity).

require that there be a “convincing proof” of the fact that $f_{\mathcal{L}}(x) = 0$. This is in contrast to the definition of IPs for the *function* $f_{\mathcal{L}}$ (Definition 3.1), for which the completeness requirement insists that for *every* input x (even those for which $f_{\mathcal{L}}(x) = 0$), there be a prover strategy that convinces the verifier of the value of $f(x)$.

The motivation behind the above formalization of IPs for languages is as follows. One may think of inputs in the language \mathcal{L} as *true statements*, and inputs not in the language as *false statements*. The above completeness and soundness properties require that all true statements have convincing proofs, and all false statements do not have convincing proofs. It is natural *not* to require that false statements have convincing refutations (i.e., convincing proofs of their falsity).

While the notions of interactive proofs for languages and functions are different, they are related in the following sense: given a *function* f , an interactive proof for f is equivalent to an interactive proof for the *language* $\mathcal{L}_f := \{(x, y) : y = f(x)\}$.

As indicated above, in this manuscript we will primarily be concerned with interactive proofs for functions instead of languages. We only talk about interactive proofs for languages when referring to complexity classes such as **NP** and **IP**, defined next.

NP and IP. Let **IP** be the class of all languages solvable by an interactive proof system with a polynomial time verifier. The class **IP** can be viewed as an interactive, randomized variant of the classical complexity class **NP** (**NP** is the class obtained from **IP** by restricting the proof system to be non-interactive and deterministic, meaning that the completeness and soundness errors are 0).

We will see soon that the class **IP** is in fact equal to **PSPACE**, the class of all languages solvable by algorithms using polynomial space (and possibly exponential time). **PSPACE** is believed to be a vastly bigger class of languages than **NP**, so this is one formalization of the statement that “interactive proofs are far more powerful than classical static (i.e, **NP**) proofs”.

By Your Powers Combined, I am IP. The key to the power of interactive proofs is the *combination* of randomness and interaction. If randomness is disallowed (equivalently, if perfect soundness $\delta_s = 0$ is required), then interaction is pointless, because the prover can predict the verifier’s messages with certainty, and hence there is no reason for the verifier to send the messages to the prover. In more detail, the proof system can be rendered non-interactive by demanding that the (non-interactive) prover send a transcript of the interactive protocol that would cause the (interactive) verifier to accept, and the (non-interactive) verifier can check that indeed the (interactive) verifier would have accepted this transcript. By perfect soundness of the interactive protocol, this non-interactive proof system is perfectly sound.

On the other hand if no interaction is allowed, but the verifier is allowed to toss random coins and accept an incorrect proof with small probability, the resulting complexity class is known as **MA** (short for *Merlin-Arthur*). This class is widely believed to be equal to **NP** (see for example [IW97]), which as stated above is believed by many researchers to be a much smaller class of problems than **IP = PSPACE**.²⁴

²⁴More precisely, it is widely believed that for every non-interactive randomized proof system $(\mathcal{V}, \mathcal{P})$ for a language \mathcal{L} , there is a non-interactive deterministic proof system $(\mathcal{V}', \mathcal{P}')$ for \mathcal{L} in which the runtime of the deterministic verifier \mathcal{V}' is at most polynomially larger than that of the randomized verifier \mathcal{V} . This would not necessarily mean that the deterministic verifier \mathcal{V}' is *just as fast* as the randomized verifier \mathcal{V} . See for example Freivald’s non-interactive randomized proof system for matrix multiplication in Section 2.2—the verifier there runs in $O(n^2)$ time, which is faster than any known deterministic verifier for the same problem, but “only” by a factor of about $O(n^{0.3728639})$, which is a (small) polynomial in the input size. This is in contrast to the transformation of the preceding paragraph from deterministic interactive proofs to non-interactive proofs, which introduces no overhead for either the verifier or the prover.

3.4 Schwartz-Zippel Lemma

Terminology. For an m -variate polynomial g , the degree of a term of g is the sum of the exponents of the variables in the term. For example if $g(x_1, x_2) = 7x_1^2x_2 + 6x_2^4$, then the degree of the term $7x_1^2x_2$ is 3, and the degree of the term $6x_2^4$ is 4. The total degree of g is the maximum of the degree of any term of g , which in the preceding example is 4.

The Lemma Itself. Interactive proofs frequently exploit the following basic property of polynomials, which is commonly known as the Schwartz-Zippel lemma [Sch80, Zip79].

Lemma 3.3 (Schwartz-Zippel Lemma). *Let \mathbb{F} be any field, and let $g : \mathbb{F}^m \rightarrow \mathbb{F}$ be a nonzero polynomial of total degree at most d . Then on any finite set $S \subseteq \mathbb{F}$,*

$$\Pr_{x \leftarrow S^m} [g(x) = 0] \leq d/|S|.$$

In words, if x is chosen uniformly at random from S^m , then the probability that $g(x) = 0$ is at most $d/|S|$. In particular, any two distinct polynomials of total degree at most d can agree on at most $d/|S|$ fraction of points in S^m .

We will not prove the lemma above, but it is easy to find a proof online (see, e.g., the wikipedia article on the lemma, or an alternative proof due to Moshkovitz [Mos10]). An easy implication of the Schwartz-Zippel lemma is that for any two distinct m -variate polynomials p and q of total degree at most d over \mathbb{F} , $p(x) = q(x)$ for at most a $d/|\mathbb{F}|$ fraction of inputs. Section 2.1.1 on Reed-Solomon fingerprinting exploited precisely this implication in the special case of univariate polynomials (i.e., $m = 1$).

3.5 Low Degree and Multilinear Extensions

Let \mathbb{F} be any finite field, and let $f : \{0, 1\}^v \rightarrow \mathbb{F}$ be any function mapping the v -dimensional Boolean hypercube to \mathbb{F} . A v -variate polynomial g over \mathbb{F} is said to be an *extension* of f if g agrees with f at all Boolean-valued inputs, i.e., $g(x) = f(x)$ for all $x \in \{0, 1\}^v$.²⁵

Preview: Why low-degree extensions are useful. One can think of a (low-degree extension g of f as an error-correcting (or, at least, *distance-amplifying*) encoding of f in the following sense: if two Boolean functions f, f' disagree at even a single input, then any degree d extensions g, g' must differ *almost everywhere* (assuming $d \ll |\mathbb{F}|$).²⁶ This is made precise by the Schwartz-Zippel lemma above, which guarantees that g and g' agree on at most $d/|\mathbb{F}|$ fraction of points in \mathbb{F}^v . This is entirely analogous to Reed-Solomon fingerprinting, which exploited the special case of Schwartz-Zippel for univariate polynomials. As we will

²⁵Later in this manuscript, we will consider extensions of functions $f : \{0, \dots, M-1\}^v \rightarrow \mathbb{F}$ with $1 < M < |\mathbb{F}|$. In this case, we say that $g : \mathbb{F}^v \rightarrow \mathbb{F}$ extends f if $g(x) = f(x)$ for all $x \in \{0, \dots, M-1\}^v$. Here, we interpret each number in $\{0, \dots, M-1\}$ as elements of \mathbb{F} via any efficiently computable injection from $\{0, \dots, M-1\}$ to \mathbb{F} .

²⁶Precisely how small d must be for a degree- d extension polynomial g to be called “low-degree” is deliberately left vague and may be context-dependent. At a minimum, d should be less than $|\mathbb{F}|$ to ensure that the probability $d/|\mathbb{F}|$ appearing in the Schwartz-Zippel lemma is less than 1; otherwise, the Schwartz-Zippel lemma is vacuous. When a low-degree extension g is used in interactive proofs or arguments, various costs of the protocol (such as proof size, verifier time, or prover time) often depend on the degree d of g , and the smaller d is, the lower these costs are.

see throughout this survey, these distance-amplifying properties give the verifier surprising power over the prover.²⁷

Definition 3.4. A multivariate polynomial g is *multilinear* if the degree of the polynomial in each variable is at most one.

For example, the polynomial $g(x_1, x_2) = x_1x_2 + 4x_1 + 3x_2$ is multilinear, but the polynomial $h(x_1, x_2) = x_2^2 + 4x_1 + 3x_2$ is not.

Throughout this survey, we will frequently use the following fact.

Fact 3.5. Any function $f: \{0, 1\}^v \rightarrow \mathbb{F}$ has a unique multilinear extension (MLE) over \mathbb{F} , and we reserve the notation \tilde{f} for this special extension of f .

That is, \tilde{f} , is the unique multilinear polynomial over \mathbb{F} satisfying $\tilde{f}(x) = f(x)$ for all $x \in \{0, 1\}^v$. See Figure 3.2 for an example of a function and its multilinear extension.

Proof of Fact 3.5. Lemma 3.6 below demonstrates that for any function $f: \{0, 1\}^v \rightarrow \mathbb{F}$, there is some multilinear polynomial that extends f . To show uniqueness of the multilinear extension of f , we want to show that if p and q are two multilinear polynomials such that $p(x) = q(x)$ for all $x \in \{0, 1\}^v$, then $p(x) = q(x)$ for all $x \in \mathbb{F}^v$ (equivalently, we want to show that the polynomial $h := p - q$ is the identically 0 polynomial).

Observe that h is also multilinear, because it is the difference of two multilinear polynomials. Furthermore, the assumption that $p(x) = q(x)$ for all $x \in \{0, 1\}^v$ implies that $h(x) = 0$ for all $x \in \{0, 1\}^v$. We now show that any such polynomial is identically 0.

Assume that h is a multilinear polynomial that vanishes on $\{0, 1\}^v$, meaning that $h(x) = 0$ for all $x \in \{0, 1\}^v$. If h is not the identically zero polynomial, then consider any term t in h of minimal degree. h must have at least one such term since h is not identically 0. For example, if $h(x_1, x_2, x_3) = x_1x_2x_3 + 2x_1x_2$, then the term $2x_1x_2$ is of minimal degree, since it has degree 2, and h has no terms of degree 1 or 0.

Now consider the input z obtained by setting all of the variables in t to 1, and all other variables to 0 (in the example above, $z = (1, 1, 0)$). At input z , term t is nonzero because all of the variables appearing in term t are set to 1. For instance, in the example above, the term $2x_1x_2$ evaluates to 2 at input $(1, 1, 0)$.

Meanwhile, by multilinearity of h , all other terms of h contain at least one variable that is not in term t (otherwise, t would not be of minimal degree in h). Since z sets all variables not in t to 0, this means that all terms in h other than t evaluate to 0 at z . It follows that $h(z) \neq 0$ (e.g., in the example above, $h(z) = 2$).

This contradicts the assumption that $h(x) = 0$ for all $x \in \{0, 1\}^v$. We conclude that any multilinear polynomial h that vanishes on $\{0, 1\}^v$ must be identically zero, as desired. \square

While any function $f: \{0, 1\}^v \rightarrow \mathbb{F}$ has many polynomials that extend it, Fact 3.5 states that exactly one of those extension polynomials is multilinear. For example, if $f(x) = 0$ for all $x \in \{0, 1\}^v$, then the multilinear extension of f is just the 0 polynomial. But $p(x_1, \dots, x_v) = x_1 \cdot (1 - x_1)$ is one example of a non-multilinear polynomial that also extends f .

MLEs have a particularly simple representation, given by Lagrange interpolation:

²⁷In fact, the use of low-degree extensions in many of the interactive proofs and arguments we describe in this survey could in principle be replaced with different error-correcting encodings that do not correspond to low-degree polynomials (see for example [Mei13, RR19] for papers in this direction). However, we will see that low-degree extensions have nice structure that enables the prover and verifier to run especially efficiently when we use low-degree extensions rather than general error-correcting encodings. It remains an important direction for future research to obtain IPs and arguments with similar (or better!) efficiency by using non-polynomial encodings.

	0	1
0	1	2
1	1	4

Figure 3.1: All evaluations of a function f mapping $\{0, 1\}^2$ to the field \mathbb{F}_5 .

	0	1	2	3	4
0	1	2	3	4	5
1	1	4	2	0	3
2	1	1	1	1	1
3	1	3	0	2	4
4	1	0	4	3	2

Figure 3.2: All evaluations of the multilinear extension, \tilde{f} of f over \mathbb{F}_5 . Via Lagrange interpolation (Lemma 3.6), $\tilde{f}(x_1, x_2) = (1 - x_1)(1 - x_2) + 2(1 - x_1)x_2 + x_1(1 - x_2) + 4x_1x_2$.

Lemma 3.6. *Let $f: \{0, 1\}^v \rightarrow \mathbb{F}$ be any function and \tilde{f} denote its multilinear extension. Then, as formal polynomials,*

$$\tilde{f}(x_1, \dots, x_v) = \sum_{w \in \{0, 1\}^v} f(w) \cdot \chi_w(x_1, \dots, x_v), \quad (3.1)$$

where, for any $w = (w_1, \dots, w_v)$,

$$\chi_w(x_1, \dots, x_v) := \prod_{i=1}^v (x_i w_i + (1 - x_i)(1 - w_i)). \quad (3.2)$$

Proof. For any vector $w \in \{0, 1\}^v$, χ_w satisfies $\chi_w(w) = 1$, and $\chi_w(y) = 0$ for all other vectors $y \in \{0, 1\}^v$. Hence, it is easy to check that $\sum_{w \in \{0, 1\}^v} f(w) \cdot \chi_w(y) = f(y)$ for all Boolean vectors $y \in \{0, 1\}^v$. In addition, the right hand side of Equation (3.1) is clearly a multilinear polynomial in (x_1, \dots, x_v) . Putting these two statements together, the right hand side of Equation (3.1) is a multilinear polynomial extending f .

The proof of Fact 3.5 established that there is only one multilinear polynomial extending f . Hence, the right hand side of Equation (3.1) must equal the unique multilinear extension \tilde{f} of f . \square

Suppose that the verifier is given as input the values $f(w)$ for all $n = 2^v$ Boolean vectors $w \in \{0, 1\}^v$. Equation (3.1) yields two efficient methods for evaluating \tilde{f} at any point $r \in \mathbb{F}^v$. The first method was described in [CTY11]: it requires $O(n \log n)$ time, and allows \mathcal{V} to make a single streaming pass over the $f(w)$ values while storing just $v + 1 = O(\log n)$ field elements. The second method is due to Vu et al. [VSBW13]: it shaves a logarithmic factor off of \mathcal{V} 's runtime, bringing it down to linear time, i.e., $O(n)$, but increases \mathcal{V} 's space usage to $O(n)$.

Lemma 3.7 ([CTY11]). *Fix a positive integer v and let $n = 2^v$. Given as input $f(w)$ for all $w \in \{0, 1\}^v$ and a vector $r \in \mathbb{F}^{\log n}$, \mathcal{V} can compute $\tilde{f}(r)$ in $O(n \log n)$ time and $O(\log n)$ words of space²⁸ with a single streaming pass over the input (regardless of the order in which the $f(w)$ value are presented).*

²⁸A “word of space” refers to the amount of data processed by a machine in one step. It is often 64 bits on modern processors. For simplicity, we assume throughout that a field element can be stored using a constant number of machine words.

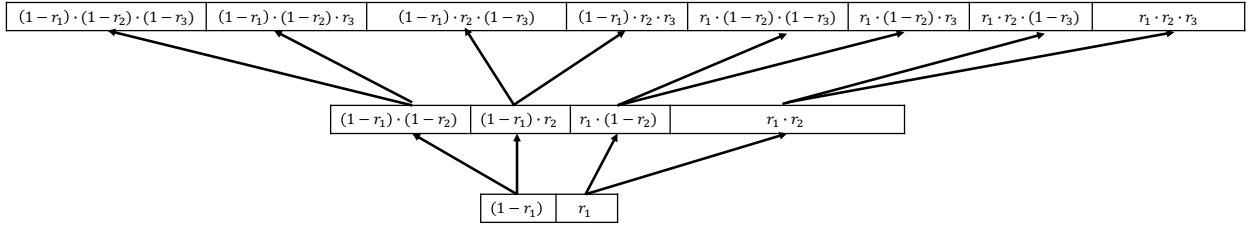


Figure 3.3: Evaluating all eight three-variate Lagrange basis polynomials at input $r = (r_1, r_2, r_3) \in \mathbb{F}^3$ via the memoization procedure in the proof of Lemma 3.8. The algorithm uses 12 field multiplications in total. In contrast, the algorithm given in Lemma 3.7 independently evaluates each Lagrange basis polynomial at r independently. This requires 2 field multiplications per basis polynomial, or $8 \cdot 2 = 16$ multiplication in total.

Proof. \mathcal{V} can compute the right hand side of Equation (3.1) incrementally from the stream by initializing $\tilde{f}(r) \leftarrow 0$, and processing each update $(w, f(w))$ via:

$$\tilde{f}(r) \leftarrow \tilde{f}(r) + f(w) \cdot \chi_w(r).$$

\mathcal{V} only needs to store $\tilde{f}(r)$ and r , which requires $O(\log n)$ words of memory (one for each entry of r). Moreover, for any w , $\chi_w(r)$ can be computed in $O(\log n)$ field operations (see Equation (3.2)), and thus \mathcal{V} can compute $\tilde{f}(r)$ with one pass over the stream, using $O(\log n)$ words of space and $O(\log n)$ field operations per update. \square

The algorithm of Lemma 3.7 computes $\tilde{f}(r)$ by evaluating each term on the right hand side of Equation (3.1) independently in $O(v)$ time and summing the results. This results in a total runtime of $O(v \cdot 2^v)$. The following lemma gives an even faster algorithm, running in time $O(2^v)$. Its speedup relative to Lemma 3.7 is obtained by *not* treating each term of the sum independently. Rather, using dynamic programming, Lemma 3.8 computes $\chi_w(r)$ for all 2^v vectors $w \in \{0, 1\}^v$ in time $O(2^v)$.

Lemma 3.8 ([VSBW13]). *Fix a positive integer v , and let $n = 2^v$. Given as input $f(w)$ for all $w \in \{0, 1\}^v$ and a vector $r = (r_1, \dots, r_v) \in \mathbb{F}^{\log n}$, \mathcal{V} can compute $\tilde{f}(r)$ in $O(n)$ time and $O(n)$ space.*

Proof. Notice the right hand side of Equation (3.1) expresses $\tilde{f}(r)$ as the inner product of two n -dimensional vectors, where (associating $\{0, 1\}^v$ and $\{0, \dots, 2^v - 1\}$ in the natural way) the w 'th entry of the first vector is $f(w)$ and the w 'th entry of the second vector is $\chi_w(r)$. This inner product can be computed in $O(n)$ time given a table of size n whose w th entry contains the quantity $\chi_w(r)$. Vu et al. show how to build such a table in time $O(n)$ using memoization.

The memoization procedure consists of $v = \log n$ stages, where Stage j constructs a table $A^{(j)}$ of size 2^j , such that for any $(w_1, \dots, w_j) \in \{0, 1\}^j$, $A^{(j)}[(w_1, \dots, w_j)] = \prod_{i=1}^j \chi_{w_i}(r_i)$. Notice $A^{(j)}[(w_1, \dots, w_j)] = A^{(j-1)}[(w_1, \dots, w_{j-1})] \cdot (w_j r_j + (1 - w_j)(1 - r_j))$, and so the j th stage of the memoization procedure requires time $O(2^j)$. The total time across all $\log n$ stages is therefore $O(\sum_{j=1}^{\log n} 2^j) = O(2^{\log n}) = O(n)$. An example of this memoization procedure for $v = 3$ is given in Figure 3.3. \square

3.6 Exercises

Exercise 3.1. Let A, B, C be $n \times n$ matrices over a field \mathbb{F} . In Section 2.2, we presented a randomized algorithm for checking that $C = A \cdot B$. The algorithm picked a random field element r , let $x = (r, r^2, \dots, r^n)$,

and output EQUAL if $Cx = A \cdot (Bx)$, and output NOT-EQUAL otherwise. Suppose instead that each entry of the vector x is chosen independently and uniformly at random from \mathbb{F} . Show that:

- If $C_{ij} = (AB)_{ij}$ for all $i = 1, \dots, n, j = 1, \dots, n$, then the algorithm outputs EQUAL for every possible choice of x .
- If there is even one $(i, j) \in [n] \times [n]$ such that $C_{ij} \neq (AB)_{ij}$, then the algorithm outputs NOT-EQUAL with probability at least $1 - 1/|\mathbb{F}|$.

Exercise 3.2. In Section 2.1, we described a communication protocol of logarithmic cost for determining whether Alice's and Bob's input vectors are equal. Specifically, Alice and Bob interpreted their inputs as degree- n univariate polynomials p_a and p_b , chose a random $r \in \mathbb{F}$ with $|\mathbb{F}| \gg n$, and compared $p_a(r)$ to $p_b(r)$. Give a different communication protocol in which Alice and Bob interpret their inputs as multilinear rather than univariate protocols over \mathbb{F} . How large should \mathbb{F} be to ensure that the probability Bob outputs the wrong answer is at most $1/n$? What is the communication cost in bits of this protocol?

Exercise 3.3. The communication protocol of Section 2.1 determined whether two vectors $a = (a_1, \dots, a_n) \in \mathbb{F}^n$ and $b = (b_1, \dots, b_n) \in \mathbb{F}^n$ are equal by interpreting a and b as specifying the *coefficients* of polynomials p_a and p_b over a finite field \mathbb{F} , i.e., $p_a(r) = \sum_{i=1}^n a_i r^i$.

- (Part a) How fast can Alice compute $p_a(r)$?
- (Part b) Let $\mathbb{F} = \mathbb{F}_p$ be a field of prime order $p \geq n$. Show that for any sequence of values $a = (a_1, \dots, a_n) \in \mathbb{F}^n$, there is a unique univariate polynomial q_a over \mathbb{F} of degree at most $n - 1$ such that $q_a(i) = a_i$ for $i = 1, \dots, n$. q_a is sometimes referred to as the *univariate low-degree extension* of the vector a . Hint: To establish the existence of such a polynomial, consider Lagrange interpolation (presented for multilinear rather than univariate polynomials in Lemma 3.6). To establish uniqueness, consider Fact 2.1.
- (Part c) Suppose rather than sending r and $p_a(r)$ to Bob, Alice sends r and $q_a(r)$ to Bob, and Bob checks whether $q_a(r) = q_b(r)$. What is the error probability of this protocol?
- (Part d) How fast can Alice compute $q_a(r)$?

Exercise 3.4. Let $p = 11$. Consider the function $f: \{0, 1\}^2 \rightarrow \mathbb{F}_p$ given by $f(0, 0) = 3, f(0, 1) = 4, f(1, 0) = 1$ and $f(1, 1) = 2$. Write out an explicit expression for the multilinear extension \tilde{f} of f . What is $\tilde{f}(2, 4)$?

Now consider the function $f: \{0, 1\}^3 \rightarrow \mathbb{F}_p$ given by $f(0, 0, 0) = 1, f(0, 1, 0) = 2, f(1, 0, 0) = 3, f(1, 1, 0) = 4, f(0, 0, 1) = 5, f(0, 1, 1) = 6, f(1, 0, 1) = 7, f(1, 1, 1) = 8$. What is $\tilde{f}(2, 4, 6)$? How many field multiplications did you perform during the calculation? Can you work through a calculation of $\tilde{f}(2, 4, 6)$ that uses "just" 20 multiplication operations? Hint: see Lemma 3.8.

Exercise 3.5. Fix some prime p of your choosing. Write a Python program that takes as input an array of length 2^ℓ specifying all evaluations of a function $f: \{0, 1\}^\ell \rightarrow \mathbb{F}_p$ and a vector $\mathbf{r} \in \mathbb{F}_p^\ell$, and outputs $\tilde{f}(\mathbf{r})$.

Chapter 4

Interactive Proofs

The first interactive proof that we cover is the sum-check protocol, due to Lund, Fortnow, Karloff, and Nisan [LFKN92]. The sum-check protocol has served as the single most important “hammer” in the design of efficient interactive proofs. Indeed, after introducing the sum-check protocol in Section 4.1, the remaining sections of this chapter apply the protocol in clean (but non-trivial) ways to solve a variety of important problems.

4.1 The Sum-Check Protocol

Suppose we are given a v -variate polynomial g defined over a finite field \mathbb{F} . The purpose of the sum-check protocol is to compute the sum:

$$H := \sum_{b_1 \in \{0,1\}} \sum_{b_2 \in \{0,1\}} \dots \sum_{b_v \in \{0,1\}} g(b_1, \dots, b_v). \quad (4.1)$$

Summing up the evaluations of a polynomial over all Boolean inputs may seem like a contrived task with limited practical utility. But to the contrary, later sections of this chapter will show that many natural problems can be directly cast as an instance of Equation (4.1).

Remark 4.1. In full generality, the sum-check protocol can compute the sum $\sum_{b \in B} g(b)$ for any $B \subseteq \mathbb{F}$, but most of the applications covered in this survey will only require $B = \{0, 1\}$.

What does the verifier gain by using the sum-check protocol? The verifier could clearly compute H via Equation (4.1) on her own by evaluating g at 2^v inputs (namely, all inputs in $\{0, 1\}^v$), but we are thinking of 2^v as an unacceptably large runtime for the verifier. Using the sum-check protocol, the verifier’s runtime will be

$$O(v + [\text{the cost to evaluate } g \text{ at a single input in } \mathbb{F}^v]).$$

This is much better than the 2^v evaluations of g required to compute H unassisted.

It also turns out that the prover in the sum-check protocol can compute all of its prescribed messages by evaluating g at $O(2^v)$ inputs in \mathbb{F}^v . This is only a constant factor more than what is required simply to compute H without proving correctness.

For presentation purposes, we assume for the rest of this section that the verifier has oracle access to g , i.e., \mathcal{V} can evaluate $g(r_1, \dots, r_v)$ for a randomly chosen vector $(r_1, \dots, r_v) \in \mathbb{F}^v$ with a single query to an

oracle.²⁹ A self-contained description of the sum-check protocol is provided in the codebox below. This is followed by a more intuitive, recursive description of the protocol.

Description of Sum-Check Protocol.

- At the start of the protocol, the prover sends a value C_1 claimed to equal the value H defined in Equation (4.1).
- In the first round, \mathcal{P} sends the univariate polynomial $g_1(X_1)$ claimed to equal

$$\sum_{(x_2, \dots, x_v) \in \{0,1\}^{v-1}} g(X_1, x_2, \dots, x_v).$$

\mathcal{V} checks that

$$C_1 = g_1(0) + g_1(1),$$

and that g_1 is a univariate polynomial of degree at most $\deg_1(g)$, rejecting if not. Here, $\deg_j(g)$ denotes the degree of $g(X_1, \dots, X_v)$ in variable X_j .

- \mathcal{V} chooses a random element $r_1 \in \mathbb{F}$, and sends r_1 to \mathcal{P} .
- In the j th round, for $1 < j < v$, \mathcal{P} sends to \mathcal{V} a univariate polynomial $g_j(X_j)$ claimed to equal

$$\sum_{(x_{j+1}, \dots, x_v) \in \{0,1\}^{v-j}} g(r_1, \dots, r_{j-1}, X_j, x_{j+1}, \dots, x_v).$$

\mathcal{V} checks that g_j is a univariate polynomial of degree at most $\deg_j(g)$, and that $g_{j-1}(r_{j-1}) = g_j(0) + g_j(1)$, rejecting if not.

- \mathcal{V} chooses a random element $r_j \in \mathbb{F}$, and sends r_j to \mathcal{P} .
- In Round v , \mathcal{P} sends to \mathcal{V} a univariate polynomial $g_v(X_v)$ claimed to equal

$$g(r_1, \dots, r_{v-1}, X_v).$$

\mathcal{V} checks that g_v is a univariate polynomial of degree at most $\deg_v(g)$, rejecting if not, and also checks that $g_{v-1}(r_{v-1}) = g_v(0) + g_v(1)$.

- \mathcal{V} chooses a random element $r_v \in \mathbb{F}$ and evaluates $g(r_1, \dots, r_v)$ with a single oracle query to g . \mathcal{V} checks that $g_v(r_v) = g(r_1, \dots, r_v)$, rejecting if not.
- If \mathcal{V} has not yet rejected, \mathcal{V} halts and accepts.

Description of the Start of the Protocol. At the start of the sum-check protocol, the prover sends a value C_1 claimed to equal the true answer (i.e., the quantity H defined in Equation (4.1)). The sum-check protocol proceeds in v rounds, one for each variable of g . At the start of the first round, the prover sends a polynomial $g_1(X_1)$ claimed to equal the polynomial $s_1(X_1)$ defined as follows:

$$s_1(X_1) := \sum_{(x_2, \dots, x_v) \in \{0,1\}^{v-1}} g(X_1, x_2, \dots, x_v). \quad (4.2)$$

$s_1(X_1)$ is defined to ensure that

$$H = s_1(0) + s_1(1). \quad (4.3)$$

²⁹This will not be the case in the applications described in later sections of this chapter. In our applications, \mathcal{V} will either be able to efficiently evaluate $g(r_1, \dots, r_v)$ unaided, or if this is not the case, \mathcal{V} will ask the prover to *tell her* $g(r_1, \dots, r_v)$, and \mathcal{P} will subsequently prove this claim is correct via further applications of the sum-check protocol.

Accordingly, the verifier checks that $C_1 = g_1(0) + g_1(1)$, i.e., the verifier checks that g_1 and the claimed answer C_1 are consistent with Equation (4.3).

Throughout, let $\deg_i(g)$ denote the degree of variable i in g . If the prover is honest, the polynomial $g_1(X_1)$ has degree $\deg_1(g)$. Hence g_1 can be specified with $\deg_1(g) + 1$ field elements, for example by sending the evaluation of g_1 at each point in the set $\{0, 1, \dots, \deg_1(g)\}$, or by specifying the $d + 1$ coefficients of g_1 .

The Rest of the Round 1. Recall that the polynomial $g_1(X_1)$ sent by the prover in round 1 is claimed to equal the polynomial $s_1(X_1)$ defined in Equation (4.2). The idea of the sum-check protocol is that \mathcal{V} will probabilistically check this equality of polynomials holds by picking a random field element $r_1 \in \mathbb{F}$, and confirming that

$$g_1(r_1) = s_1(r_1). \quad (4.4)$$

Clearly, if g_1 is as claimed, then this equality holds for all $r_1 \in \mathbb{F}$ (i.e., this probabilistic protocol for checking that $g_1 = s_1$ as formal polynomials is complete). Meanwhile, if $g_1 \neq s_1$, then with probability at least $1 - \deg_1(g)/|\mathbb{F}|$ over the verifier's choice of r_1 , Equation (4.4) fails to hold. This is because two distinct degree d univariate polynomials agree on at most d inputs. This means that this protocol for checking that $g_1 = s_1$ by checking that equality holds at a random input r_1 is sound, so long as $|\mathbb{F}| \gg \deg_1(g)$.

The remaining issue is the following: can \mathcal{V} efficiently compute both $g_1(r_1)$ and $s_1(r_1)$, in order to check that Equation (4.4) holds? Since \mathcal{P} sends \mathcal{V} an explicit description of the polynomial g_1 , it is possible for the \mathcal{V} to evaluate $g_1(r_1)$ in $O(d_1)$ time. In contrast, evaluating $s_1(r_1)$ is not an easy task for \mathcal{V} , as s_1 is defined as a sum over 2^{v-1} evaluations of g . This is only a factor of two smaller than the number of terms in the sum defining H (Equation (4.1)). Fortunately, Equation (4.4) expresses s_1 as the sum of the evaluations of a $(v-1)$ -variate polynomial over the Boolean hypercube (the polynomial being $g(r_1, X_2, \dots, X_v)$ that is defined over the variables X_2, \dots, X_v). This is exactly the type of expression that the sum-check protocol is designed to check. Hence, rather than evaluating $s_1(r_1)$ on her own, \mathcal{V} instead *recursively* applies the sum-check protocol to evaluate $s_1(r_1)$.

Recursive Description of Rounds 2, ..., v. The protocol thus proceeds in this recursive manner, with one round per recursive call. This means that in round j , variable X_j gets *bound* to a random field element r_j chosen by the verifier. This process proceeds until round v , in which the prover is forced to send a polynomial $g_v(X_v)$ claimed to equal $s_v := g(r_1, \dots, r_{v-1}, X_v)$. When the verifier goes to check that $g_v(r_v) = s_v(r_v)$, there is no need for further recursion: since the verifier is given oracle access to g , \mathcal{V} can evaluate $s_v(r_v) = g(r_1, \dots, r_v)$ with a single oracle query to g .

Iterative Description of the Protocol. Unpacking the recursion described above, here is an equivalent description of what happens in round j of the sum-check protocol. At the start of round j , variables X_1, \dots, X_{j-1} have already been bound to random field elements r_1, \dots, r_{j-1} , and \mathcal{V} chooses a value r_{j-1} uniformly at random from \mathbb{F} and sends r_{j-1} to \mathcal{P} . In return, the prover sends a polynomial $g_j(X_j)$, and claims that

$$g_j(X_j) = \sum_{(x_{j+1}, \dots, x_v) \in \{0, 1\}^{v-j}} g(r_1, \dots, r_{j-1}, X_j, x_{j+1}, \dots, x_v). \quad (4.5)$$

The verifier compares the two most recent polynomials by checking $g_{j-1}(r_{j-1}) = g_j(0) + g_j(1)$, and rejecting otherwise. The verifier also rejects if the degree of g_j is too high: each g_j should have degree at most $\deg_j(g)$, the degree of variable x_j in g .

In the final round, the prover has sent $g_v(X_v)$ which is claimed to be $g(r_1, \dots, r_{v-1}, X_v)$. \mathcal{V} now checks that $g_v(r_v) = g(r_1, \dots, r_v)$ (recall that we assumed \mathcal{V} has oracle access to g). If this check succeeds, and so do all previous checks, then the verifier is convinced that $H = g_1(0) + g_1(1)$.

The following proposition formalizes the completeness and soundness properties of the sum-check protocol.

Proposition 4.1. *Let g be a v -variate polynomial of total degree at most d in each variable, defined over a finite field \mathbb{F} . For any specified $H \in \mathbb{F}$, let \mathcal{L} be the language of polynomials g (given as an oracle) such that*

$$H = \sum_{b_1 \in \{0,1\}} \sum_{b_2 \in \{0,1\}} \dots \sum_{b_v \in \{0,1\}} g(b_1, \dots, b_v).$$

The sum-check protocol is an interactive proof system for \mathcal{L} with completeness error $\delta_c = 0$ and soundness error $\delta_s \leq vd/|\mathbb{F}|$.

Proof. Completeness is evident: if the prover sends the prescribed polynomial $g_j(X_j)$ at all rounds j , then \mathcal{V} will accept with probability 1. We offer two proofs of soundness, the first of which reasons in a manner analogous to the recursive description of the protocol, and the second of which reasons in a manner analogous to the iterative description.

Inductive Proof of Soundness. One way to prove soundness is by induction on v . In the case $v = 1$, \mathcal{P} 's only message specifies a degree d univariate polynomial $g_1(X_1)$. If $g_1(X_1) \neq g(X_1)$, then because any two distinct degree d univariate polynomials can agree at most d inputs, $g_1(r_1) \neq g(r_1)$ with probability at least $1 - d/|\mathbb{F}|$ over the choice of r_1 , and hence \mathcal{V} 's final check will cause \mathcal{V} to reject with probability at least $1 - d/|\mathbb{F}|$.

Assume by way of induction that for all $v - 1$ -variate polynomials, the sum-check protocol has soundness error at most $(v - 1)d/|\mathbb{F}|$. Let $s_1(X_1) = \sum_{x_2, \dots, x_v \in \{0,1\}^{v-1}} g(X_1, x_2, \dots, x_v)$. Suppose \mathcal{P} sends a polynomial $g_1(X_1) \neq s_1(X_1)$ in Round 1. Then because any two distinct degree d univariate polynomials can agree at most d inputs, $g_1(r_1) \neq s_1(r_1)$ with probability at least $1 - d/|\mathbb{F}|$. Conditioned on this event, \mathcal{P} is left to prove the false claim in Round 2 that $g_1(r_1) = \sum_{(x_2, \dots, x_v) \in \{0,1\}^{v-1}} g(r_1, x_2, \dots, x_v)$. Since $g(r_1, x_2, \dots, x_v)$ is a $(v - 1)$ -variate polynomial of total degree d , the inductive hypothesis implies that \mathcal{V} will reject at some subsequent round of the protocol with probability at least $1 - d(v - 1)/|\mathbb{F}|$. Therefore, \mathcal{V} will reject with probability at least

$$\begin{aligned} & 1 - \Pr[s_1(r_1) \neq g_1(r_1)] - (1 - \Pr[\mathcal{V} \text{ rejects in some Round } j > 1 | s_1(r_1) \neq g_1(r_1)]) \\ & \geq 1 - \frac{d}{|\mathbb{F}|} - \frac{d(v-1)}{|\mathbb{F}|} = 1 - \frac{dv}{|\mathbb{F}|}. \end{aligned}$$

Non-Inductive Proof of Soundness. A second way to prove soundness conceptually follows the iterative description of the sum-check protocol. Specifically, if $H \neq \sum_{(x_1, \dots, x_v) \in \{0,1\}^v} g(x_1, x_2, \dots, x_v)$, then the only way the prover can convince the verifier to accept is if there is at least one round i such that the prover sends a univariate polynomial $g_i(X_i)$ that does not equal the prescribed polynomial

$$s_i(X_i) = \sum_{(x_{i+1}, \dots, x_v) \in \{0,1\}^{v-i}} g(r_1, r_2, \dots, r_{i-1}, X_i, x_{i+1}, \dots, x_v),$$

and yet $g_i(r_i) = s_i(r_i)$. For every round i , g_i and s_i both have degree at most d , and hence if $g_i \neq s_i$, the probability that $g_i(r_i) = s_i(r_i)$ is at most $d/|\mathbb{F}|$. By a union bound over all v rounds, we may conclude that the probability that there is any round i such that the prover sends a polynomial $g_i \neq s_i$ yet $g_i(r_i) = s_i(r_i)$ is at most $dv/|\mathbb{F}|$. □

Communication	Rounds	\mathcal{V} time	\mathcal{P} time
$O(\sum_{i=1}^v \deg_i(g))$ field elements	v	$O(v + \sum_{i=1}^v \deg_i(g)) + T$	$O(\sum_{i=1}^v \deg_i(g) \cdot 2^{v-i} \cdot T)$ $= O(2^v \cdot T)$ if $\deg_i(g) = O(1)$ for all i

Table 4.1: Costs of the sum-check protocol when applied to a v -variate polynomial g over \mathbb{F} . Here, $\deg_i(g)$ denotes the degree of variable i in g , and T denotes the cost of an oracle query to g .

Discussion of costs. There is one round in the sum-check protocol for each of the v variables of g . The total prover-to-verifier communication is $\sum_{i=1}^v (\deg_i(g) + 1) = v + \sum_{i=1}^v \deg_i(g)$ field elements, and the total verifier-to-prover communication is v field elements (one per round). In particular, if $\deg_i(g) = O(1)$ for all j , then the communication cost is $O(v)$ field elements.³⁰

The running time of the verifier over the entire execution of the protocol is proportional to the total communication, plus the cost of a single oracle query to g to compute $g(r_1, \dots, r_v)$.

Determining the running time of the prover is less straightforward. Recall that \mathcal{P} can specify g_j by sending for each $i \in \{0, \dots, \deg_j(g)\}$ the value:

$$g_j(i) = \sum_{(x_{j+1}, \dots, x_v) \in \{0,1\}^{v-j}} g(r_1, \dots, r_{j-1}, i, x_{j+1}, \dots, x_v). \quad (4.6)$$

An important insight is that the number of terms defining the value $g_j(i)$ in Equation (4.6) falls geometrically with j : in the j th sum, there are only 2^{v-j} terms, each corresponding to a Boolean vector in $\{0, 1\}^{v-j}$. Thus, the total number of terms that must be evaluated over the course of the protocol is $\sum_{j=1}^v \deg_j(g) 2^{v-j}$ which is at most $O(2^v)$ if $\deg_j(g) = O(1)$ for all j . Consequently, if \mathcal{P} is given oracle access to g , then \mathcal{P} will require just $O(2^v)$ time.

In all of the applications covered in this survey, \mathcal{P} will not have oracle access to the truth table of g , and the key to many of the results in this survey is to show that \mathcal{P} can nonetheless evaluate g at all of the necessary points in close to $O(2^v)$ total time.

The costs of the sum-check protocol are summarized in Table 4.1. Since \mathcal{P} and \mathcal{V} will not be given oracle access to g in applications, the table makes the number of oracle queries to g explicit.

Remark 4.2. An important feature of the sum-check protocol is that the the verifier’s messages to the prover are simply random field elements, and hence entirely independent of the input polynomial g . In fact, the only information \mathcal{V} needs about the polynomial g to execute its part of the protocol is an upper bound on the degree of g in each of its v variables, and the ability to evaluate g at a random point $r \in \mathbb{F}^v$.³¹

This means that \mathcal{V} can apply the sum-check protocol *even without knowing the polynomial g to which the protocol is being applied*, so long as \mathcal{V} knows an upper bound on the degree of the polynomial in each variable, and later obtains the ability to evaluate g at a random point $r \in \mathbb{F}^v$. In contrast, the prover does need to know the precise polynomial g in order to compute each of its messages over the course of the sum-check protocol.

Preview: Why multilinear extensions are useful: ensuring a fast prover. We will see several scenarios where it is useful to compute $H = \sum_{x \in \{0,1\}^v} f(x)$ for some function $f: \{0, 1\}^v \rightarrow \mathbb{F}$ derived from the verifier’s

³⁰In practical applications of the sum-check protocol, \mathbb{F} will often be a field of size between 2^{128} and 2^{256} , meaning that any field element can be specified with between 16 and 32 bytes. These field sizes are large enough to ensure very low soundness error of the sum-check protocol, while being small enough that field operations remain fast.

³¹And $g(r)$ is needed by \mathcal{V} only in order for the verifier to perform its final check of the prover’s final message in the protocol. All other checks that \mathcal{V} performs on the messages sent by \mathcal{P} can be performed with no knowledge of g .

input. We can compute H by applying the sum-check protocol to any low-degree extension g of f . When $g = \tilde{f}$, or g is itself a product of a small number of multilinear polynomials, then the prover in the sum-check protocol applied to g can be implemented extremely efficiently. Specifically, as we show later in Lemma 4.5, Lemma 3.6 (which gave an explicit expression for \tilde{f} in terms of Lagrange basis polynomials) can be exploited to ensure that enormous cancellations occur in the computation of the prover's messages, allowing fast computation.

Preview: Why using multilinear extensions is not always possible: Ensuring a fast verifier. Although the use of the MLE \tilde{f} typically ensures fast computation for the prover, \tilde{f} cannot be used in all applications. The reason is that the verifier has to be able to evaluate \tilde{f} at a random point $r \in \mathbb{F}^v$ to perform the final check in the sum-check protocol, and in some settings, this computation would be too costly.

Lemma 3.8 gives a way for \mathcal{V} to evaluate $\tilde{f}(r)$ in time $O(2^v)$, given all evaluations of f at Boolean inputs. This might or might not be an acceptable runtime, depending on the relationship between v and the verifier's input size n . If $v \leq \log_2 n + O(\log \log n)$, then $O(2^v) = \tilde{O}(n)$,³² and the verifier runs in quasilinear time³³. But we will see some applications where $v = c \log n$ for some constant $c > 1$, and others where $v = n$ (see, e.g., the #SAT protocol in Section 4.2). In these settings, $O(2^v)$ runtime for the verifier is unacceptable, and we will be forced to use an extension g of f that has a succinct representation, enabling \mathcal{V} to compute $g(r)$ in much less than 2^v time. Sometimes \tilde{f} itself has such a succinct representation, but other times we will be forced to use a higher-degree extension of f . See Exercise 4.2 and Exercise 4.3 (Parts (d) and (e)) for further details.

Example Execution of the Sum-Check Protocol. Let $g(X_1, X_2, X_3) = 2X_1^3 + X_1X_3 + X_2X_3$. The sum of g 's evaluations over the Boolean hypercube is $H = 12$. When the sum-check protocol is applied to g , the honest prover's first message in the protocol is the univariate polynomial $s_1(X_1)$ equal to:

$$\begin{aligned} &g(X_1, 0, 0) + g(X_1, 0, 1) + g(X_1, 1, 0) + g(X_1, 1, 1) \\ &= (2X_1^3) + (2X_1^3 + X_1) + (2X_1^3) + (2X_1^3 + X_1 + 1) = 8X_1^3 + 2X_1 + 1. \end{aligned}$$

The verifier checks that $s_1(0) + s_1(1) = 12$, and then sends the prover r_1 . Suppose that $r_1 = 2$. The honest prover would then respond with the univariate polynomial

$$s_2(X_2) = g(2, X_2, 0) + g(2, X_2, 1) = 16 + (16 + 2 + X_2) = 34 + X_2.$$

The verifier checks that $s_2(0) + s_2(1) = s_1(r_1)$, which amounts in this example to confirming that $34 + (34 + 1) = 8 \cdot (2^3) + 4 + 1$; indeed, both the left hand side and right hand side equal 69. The verifier then sends the prover r_2 . Suppose that $r_2 = 3$. The honest prover would respond with the univariate polynomial $s_3(X_3) = g(2, 3, X_3) = 16 + 2X_3 + 3X_3 = 16 + 5X_3$. The verifier picks a random field element r_3 and confirms that $s_3(r_3) = g(2, 3, r_3)$ by making an oracle query to g .

4.2 First Application of Sum-Check: #SAT \in IP

Let ϕ be any Boolean formula on n variables of size $S = \text{poly}(n)$.³⁴ That is, ϕ is a directed tree containing S gates, each computing an AND, OR or NOT operation, with leaves corresponding to inputs x_1, \dots, x_n

³²The notation $\tilde{O}(\cdot)$ hides polylogarithmic factors. So, for example, $n \log^4 n = \tilde{O}(n)$.

³³Quasilinear time means time $\tilde{O}(n)$; i.e., at most a polylogarithmic factor more than linear.

³⁴ $S = \text{poly}(n)$ means that S is bounded above by $O(n^k)$ for some constant $k \geq 0$.

(see Figure 4.1 for an example). The AND and OR gates have fan-in 2, while the NOT gates have fan-in 1. Abusing notation, we will use ϕ to refer to both the formula itself and the function on $\{0,1\}^n$ that it computes. In the #SAT problem, the goal is to compute the number of satisfying assignments of ϕ , or equivalently to compute $\sum_{x \in \{0,1\}^n} \phi(x)$. This is precisely the kind of function that Lund et al. [LFKN92] designed the sum-check protocol to compute.

However, in order to apply the sum-check protocol, we need to identify a polynomial extension g of ϕ of total degree $\text{poly}(S)$ over a suitable finite field \mathbb{F} . Moreover, we need the verifier to be able to evaluate g at a random point r in polynomial time. We define g as follows.

Let \mathbb{F} be a finite field of size at least, say, S^4 (in the application of the sum-check protocol below, the soundness error will be at most $2Sn/|\mathbb{F}|$, so the field should be big enough to ensure that this quantity is acceptably small. If $|\mathbb{F}| \approx S^4$, then the soundness error is at most $2/S^2$. Bigger fields will ensure even smaller soundness error).

We can turn ϕ into an *arithmetic* circuit ψ over \mathbb{F} that computes the desired extension g of ϕ .³⁵ An arithmetic circuit \mathcal{C} has input gates, output gates, intermediate gates, and directed wires between them. Each gate computes addition or multiplication over a finite field \mathbb{F} .

For any gate in ϕ computing the AND of two inputs y, z , ψ replaces $\text{AND}(y, z)$ with multiplication of y and z over \mathbb{F} . It is easy to check that the bivariate polynomial $y \cdot z$ extends the Boolean function $\text{AND}(y, z)$, i.e., $\text{AND}(y, z) = y \cdot z$ for all $y, z \in \{0, 1\}$. Likewise, ψ replaces a gate computing $\text{NOT}(y)$ by $1 - y$, and a gate computing $\text{OR}(y, z)$ by $y + z - y \cdot z$. This transformation is depicted in Figures 4.1 and 4.2. It is easy to check that $\psi(x) = \phi(x)$ for all $x \in \{0, 1\}^n$, and that the number of gates in the arithmetic circuit ψ is at most $3S$.

For the polynomial g computed by ψ , $\sum_{i=1}^v \deg_i(g) \leq 2S$.³⁶ Thus, the total communication cost of the sum-check protocol applied to g is $O(n + S)$ field elements, and \mathcal{V} requires $O(n + S)$ time in total to check the first $n - 1$ messages from \mathcal{P} . To check \mathcal{P} 's final message, \mathcal{V} must also evaluate $g(r)$ for the random point $r \in \mathbb{F}^n$ chosen during the sum-check protocol. \mathcal{V} can clearly evaluate $g(r)$ gate-by-gate in time $O(n + S)$. Since the polynomial g has n variables and its degree in each variable is at most $2S$, the soundness error of the sum-check protocol applied to g is at most $2Sn/|\mathbb{F}|$.

As explained in Section 4.1, the prover runs in time (at most) $2^n \cdot T \cdot (\sum_{i=1}^n \deg_i(g))$, where T is the cost of evaluating g at a point. Since g can be evaluated at any point in time $O(S)$ by evaluating ψ gate-by-gate, the prover in the #SAT protocol runs in time $O(S^2 \cdot 2^n)$. The costs of this protocol are summarized in Table 4.2.

IP = PSPACE. The above #SAT protocol comes quite close to establishing a famous result, namely that **IP = PSPACE** [LFKN92, Sha92].³⁷ That is, the class of problems solvable by interactive proofs with a polynomial-time verifier is exactly equal to the class of problems solvable in polynomial space.

To show that **IP** \subseteq **PSPACE**, one needs to show that for any language \mathcal{L} solvable by an interactive proof

³⁵The difference between a formula and a circuit is that gates in a circuit may have fan-out more than 1, while all gates in a formula have fan-out exactly 1.

³⁶Here is an inductive proof of this fact. It is clearly true if ϕ consists of just one gate. Suppose by way of induction that it is true when ϕ consists of at most $S - 1$ gates, and suppose that the output gate of ϕ is an AND gate (a similar argument applies if the output gate is an OR gate or NOT gate). The two in-neighbors of the output gate partition ϕ into two disjoint subformulas ϕ_1, ϕ_2 of sizes S_1, S_2 such that $S_1 + S_2 = S$ and $\phi(x) = \text{AND}(\phi_1(x), \phi_2(x))$. By the induction hypothesis, arithmetizing the two subformulas yields extension polynomials g_1, g_2 such that for $j = 1, 2$, $\sum_{i=1}^v \deg_i(g_j) \leq 2S_j$. The arithmetization of ϕ is $g = g_1 \cdot g_2$, which satisfies $\sum_{i=1}^v \deg_i(g) \leq 2S_1 + 2S_2 = 2S$.

³⁷Here, **PSPACE** is the class of decision problems that can be solved by some algorithm whose memory usage is bounded by some constant power of n , e.g., n^{10} .

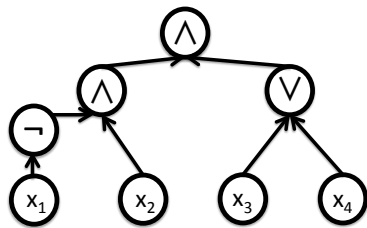


Figure 4.1: A Boolean formula ϕ .

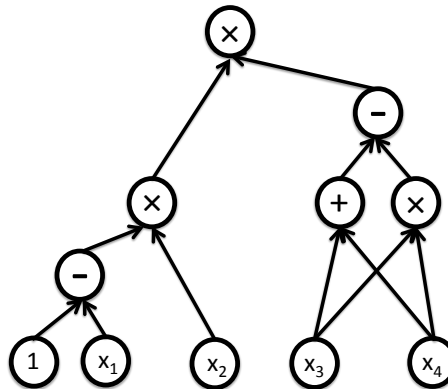


Figure 4.2: An arithmetic circuit ψ computing a polynomial extension g of ϕ over a finite field \mathbb{F} .

with communication cost at most, say, n^{10} , there is an algorithm \mathcal{A} that solves the problem in space at most, say, n^{20} . Note that the resulting algorithm might be *extremely* slow, potentially taking time *exponential* in n^{20} . That is, the inclusion $\mathbf{IP} \subseteq \mathbf{PSPACE}$ does *not* state that any problem solvable by an interactive proof with an efficient verifier necessarily has a *fast* algorithm, but does state that the problem has a reasonably small-space algorithm.

Very roughly speaking, the algorithm \mathcal{A} on input x will determine whether $x \in \mathcal{L}$ by ascertaining whether or not there is a prover strategy that causes the verifier to accept with probability at least $2/3$. It does this by actually identifying an optimal prover strategy, i.e., finding the prover that maximizes the probability the verifier accepts on input x , and determining exactly what that probability is.

In slightly more detail, it suffices to show that for any interactive proof protocol with communication cost $c(n) \geq n$, that (a) an optimal prover strategy can be computed in space $\text{poly}(c(n))$ and (b) the verifier's acceptance probability when the prover executes that optimal strategy can also be computed in space $\text{poly}(c(n))$. Together, (a) and (b) imply that $\mathbf{IP} \subseteq \mathbf{PSPACE}$ because $x \in \mathcal{L}$ if and only if the optimal prover strategy induces the verifier to accept input x with probability at least $2/3$.

Property (b) holds simply because for any fixed prover strategy \mathcal{P} and input x , the probability the verifier accepts when interacting with \mathcal{P} can be computed in space $\text{poly}(c(n))$ by enumerating over every possible setting of the verifier's random coins and computing the fraction of settings that lead the verifier to accept. Again, note that this enumeration procedure is *extremely* slow (requiring time exponential in $c(n)$), but can be done in space just $\text{poly}(c(n))$. For a proof of Property (a), the interested reader is directed to [Koz06, Lecture 17].³⁸

The more challenging direction is to show that $\mathbf{PSPACE} \subseteq \mathbf{IP}$. The #SAT protocol of Lund et al. [LFKN92] described above already contains the main ideas necessary to prove this. Shamir [Sha92] extended the #SAT protocol to solve the \mathbf{PSPACE} -complete language TQBF, and Shen [She92] gave a simpler proof. We do not cover Shamir or Shen's extensions of the #SAT protocol here, since later (Section 4.5.4), we will provide a different and quantitatively stronger proof that $\mathbf{PSPACE} \subseteq \mathbf{IP}$.

³⁸As stated in [Koz06, Lecture 17], the result that $\mathbf{IP} \subseteq \mathbf{PSPACE}$ is attributed to a manuscript Paul Feldman in a paper by Goldwasser and Siper [GS86], and also follows from the analysis in [GS86].

Communication	Rounds	\mathcal{V} time	\mathcal{P} time
$O(S)$ field elements	n	$O(S)$	$O(S^2 \cdot 2^n)$

Table 4.2: Costs of the #SAT protocol of Section 4.2 when applied to a Boolean formula $\phi : \{0, 1\}^n \rightarrow \{0, 1\}$ with S gates of fan-in at most two.

4.3 Second Application: A Simple IP for Counting Triangles in Graphs

Section 4.2 used the sum-check protocol to give an IP for the #SAT problem, in which the verifier runs in time polynomial in the input size, and the prover runs in time exponential in the input size. This may not seem particularly useful, because in the real-world an exponential-time prover simply will not scale to even moderately-sized inputs. Ideally, we want provers that run in polynomial rather than exponential time, and we want verifiers that run in *linear* rather than polynomial time. IPs achieving such time costs are often called *doubly-efficient*, with the terminology chosen to highlight that both the verifier and prover are highly efficient. The remainder of this chapter is focused on the development of doubly-efficient IPs.

As a warmup, in this section, we apply the sum-check protocol in a straightforward manner to give a simple, doubly-efficient IP for an important graph problem: counting triangles. We give an even more efficient (but less simple) IP for this problem in Section 4.5.1.

To define the problem, let $G = (V, E)$ be a simple graph on n vertices.³⁹ Here, V denotes the set of vertices of G , and E denotes the edges in G . Let $A \in \{0, 1\}^{n \times n}$ be the adjacency matrix of G , i.e., $A_{i,j} = 1$ if and only if $(i, j) \in E$. In the counting triangles problem, the input is the adjacency matrix A , and the goal is to determine the number of unordered node triples $(i, j, k) \in V \times V \times V$ such that i, j , and k are all connected to each other, i.e., (i, j) , (j, k) and (i, k) are all edges in E .

At first blush, it is totally unclear how to express the number of triangles in G as the sum of the evaluations of a degree-2 polynomial g over all inputs in $\{0, 1\}^v$, as per Equation (4.1). After all, the counting triangles problem itself makes no reference to any low-degree polynomial g , so where will g come from? This is where multilinear extensions come to the rescue.

For it to make sense to talk about multilinear extensions, we need to view the adjacency matrix A not as a matrix, but rather as a function f_A mapping $\{0, 1\}^{\log n} \times \{0, 1\}^{\log n}$ to $\{0, 1\}$. The natural way to do this is to define $f_A(x, y)$ so that it interprets x and y as the binary representations of some integers i and j between 1 and n , and outputs $A_{i,j}$. See Figure 4.3 for an example.

Then the number of triangles, Δ , in G can be written:

$$\Delta = \frac{1}{6} \sum_{x,y,z \in \{0,1\}^{\log n}} f_A(x,y) \cdot f_A(y,z) \cdot f_A(x,z). \quad (4.7)$$

To see that this equality is true, observe that the term for x, y, z in the above sum is 1 if edges (x, y) , (y, z) , and (x, z) all appear in G , and is 0 otherwise. The factor $1/6$ comes in because the sum over *ordered* node triples (i, j, k) counts each triangle 6 times, once for each permutation of i, j , and k .

Let \mathbb{F} be a finite field of size $p \geq 6n^3$, where p is a prime, and let us view all entries of A as elements of \mathbb{F} . Here, we are choosing p large enough so that 6Δ is guaranteed to be in $\{0, 1, \dots, p-1\}$, as the maximum number of triangles in any graph on n vertices is $\binom{n}{3} \leq n^3$. This ensures that, if we associate elements of \mathbb{F} with integers in $\{0, 1, \dots, p-1\}$ in the natural way, then Equation (4.7) holds even when all additions and multiplications are done in \mathbb{F} rather than over the integers. (Choosing a large field to work over has the

³⁹A simple graph is one that is undirected and unweighted, with no self-loops or repeat edges.

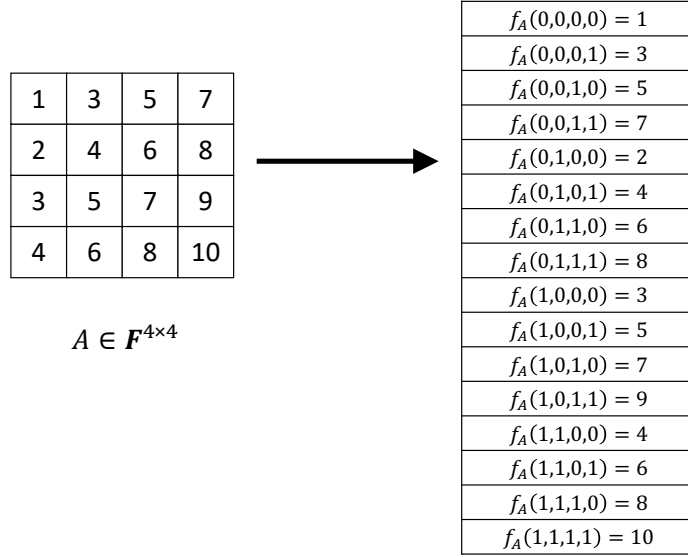


Figure 4.3: Example of how to view an $n \times n$ matrix A with entries from \mathbb{F} as a function f_A mapping the domain $\{0, 1\}^{\log_2(n)} \times \{0, 1\}^{\log_2(n)}$ to \mathbb{F} , when $n = 4$.

added benefit of ensuring good soundness error, as the soundness error of the sum-check protocol decreases linearly with field size.)

At last we are ready to describe the polynomial g to which we will apply the sum-check protocol to compute 6Δ . Recalling that \tilde{f}_A is the multilinear extension of f_A over \mathbb{F} , define the $(3 \log n)$ -variate polynomial g to be:

$$g(X, Y, Z) = \tilde{f}_A(X, Y) \cdot \tilde{f}_A(Y, Z) \cdot \tilde{f}_A(X, Z).$$

Equation (4.7) implies that:

$$6\Delta = \sum_{x,y,z \in \{0,1\}^{\log n}} g(x, y, z),$$

so applying the sum-check protocol to g yields an IP computing 6Δ .

Example. Consider the smallest non-empty graph, namely the two-vertex graph with a single undirected edge connecting the two vertices. There are no triangles in this graph. This is because there are fewer than three vertices in the entire graph, and there are no self-loops. That is, by the pigeonhole principle, for every triple of vertices (i, j, k) , at least two of the vertices are the *same* vertex (i.e., at least one of $i = j$, $j = k$, or $i = k$ holds), and since there are no self-loops in the graph, these two vertices are not connected to each other by an edge. In this example, the adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case,

$$\tilde{f}_A(a, b) = a \cdot (1 - b) + b \cdot (1 - a),$$

and g is the following 3-variate polynomial:

$$g(X, Y, Z) = (X \cdot (1 - Y) + Y \cdot (1 - X)) (Y \cdot (1 - Z) + Z \cdot (1 - Y)) (X \cdot (1 - Z) + Z \cdot (1 - X)).$$

It is not hard to see that $g(x, y, z) = 0$ for all $(x, y, z) \in \{0, 1\}^3$, and hence applying the sum-check protocol to g reveals that the number of triangles in the graph is $\frac{1}{6} \cdot \sum_{(x,y,z) \in \{0,1\}^3} g(x, y, z) = 0$.

Costs of the Protocol. Since the polynomial g is defined over $3 \log n$ variables, there are $3 \log n$ rounds. Since g has degree at most 2 in each of its $3 \log n$ variables, the total number of field elements sent by the prover in each round is at most 3. This means that the communication cost is $O(\log n)$ field elements ($9 \log n$ elements sent from prover to verifier, and at most $3 \log n$ sent from verifier to prover).

The verifier's runtime is dominated by the time to perform the final check in the sum-check protocol. This requires evaluating g at a random input $(r_1, r_2, r_3) \in \mathbb{F}^{\log n} \times \mathbb{F}^{\log n} \times \mathbb{F}^{\log n}$, which in turn requires evaluating $\tilde{f}_A(r_1, r_2)$, $\tilde{f}_A(r_2, r_3)$ and $\tilde{f}_A(r_1, r_3)$. Each of these 3 evaluations can be computed in $O(n^2)$ field operations using Lemma 3.8, which is linear in the size of the input matrix A .

The prover's runtime is clearly at most $O(n^5)$. This is because, since there are $3 \log_2 n$ rounds of the protocol, it is sufficient for the prover to evaluate g at $O(n^3)$ inputs (see Table 4.1), and as explained in the previous paragraph, g can be evaluated at any input in $\mathbb{F}^{3 \log n}$ in $O(n^2)$ time. In fact, more sophisticated algorithmic insights introduced in the next section can bring the prover runtime down to $O(n^3)$, which is competitive with the naive unverifiable algorithm for counting triangles that iterates over every triple of vertices and checks if they form a triangle. We omit further discussion of how to achieve prover time $O(n^3)$ in the protocol of this section, as Section 4.5.1 gives a different IP for counting triangles, in which the prover's runtime is much less than $O(n^3)$.

A Bird's Eye View. Hopefully the above protocol for counting triangles gives a sense of how problems that people care about in practice can be expressed as instances of Equation (4.1) in non-obvious ways. The general paradigm works as follows. An input x of length n is viewed as a function f_x mapping $\{0, 1\}^{\log n}$ to some field \mathbb{F} . And then the multilinear extension \tilde{f}_x of f_x is used in some way to construct a low-degree polynomial g such that, as per Equation (4.1), the desired answer equals the sum of the evaluations of g over the Boolean hypercube. The remaining sections of this chapter cover additional examples of this paradigm.

4.4 Third Application: Super-Efficient IP for MATMULT

This section describes a highly optimized IP protocol for matrix multiplication (MATMULT) from [Tha13]. While this MATMULT protocol is of interest in its own right, it is included here for didactic reasons: it displays, in a clean and unencumbered setting, all of the algorithmic insights that are exploited later in this survey to give more general IP and MIP protocols.

Given two $n \times n$ input matrix A, B over field \mathbb{F} , the goal of MATMULT is to compute the product matrix $C = A \cdot B$.

4.4.1 Comparison to Freivalds' Protocol

Recall from Section 2.2 that, in 1977, Freivalds [Fre77] gave the following verification protocol for MATMULT: to check that $A \cdot B = C$, \mathcal{V} picks a random vector $x \in \mathbb{F}^n$, and accepts if $A \cdot (Bx) = Cx$. \mathcal{V} can compute $A \cdot (Bx)$ with two matrix-vector multiplications, which requires just $O(n^2)$ time. Thus, in Freivalds' protocol, \mathcal{P} simply finds and sends the correct answer C , while \mathcal{V} runs in optimal $O(n^2)$ total time. Today, Freivalds' protocol is regularly covered in introductory textbooks on randomized algorithms.

At first glance, Freivalds' protocol seems to close the book on verification protocols for MATMULT, since the runtimes of both \mathcal{V} and \mathcal{P} are optimal: \mathcal{P} does *no* extra work to prove correctness of the answer matrix C , \mathcal{V} runs in time linear in the input size, and the protocol is even non-interactive (\mathcal{P} just sends the answer matrix C to \mathcal{V}).

However, there is a sense in which it is possible to improve on Freivalds’ protocol by introducing interaction between \mathcal{P} and \mathcal{V} . In many settings, algorithms invoke `MATMULT`, but they are not really interested in the full answer matrix. Rather, they apply a simple post-processing step to the answer matrix to arrive at the quantity of true interest. For example, the best-known graph diameter algorithms repeatedly square the adjacency matrix of the graph, but ultimately they are not interested in the matrix powers—they are only interested in a single number. As another example, discussed in detail in Section 4.5.1, the fastest known algorithm for counting triangles in dense graphs invokes matrix multiplication, but is ultimately only interested in a single number (namely the number of triangles in the graph).

If Freivalds’ protocol is used to verify the matrix multiplication steps of these algorithms, the actual product matrices must be sent for each step, necessitating $\Omega(n^2)$ communication. In practice, this can easily be many terabytes of data, even on graphs G with a few million nodes (also, even if G is sparse, powers of G ’s adjacency matrix may be dense).

This section describes an interactive matrix multiplication protocol from [Tha13] that preserves the runtimes of \mathcal{V} and \mathcal{P} from Freivalds’ protocol, but avoids the need for \mathcal{P} to send the full answer matrix in the settings described above—in these settings, the communication cost of the interactive protocol is just $O(\log n)$ field elements per matrix multiplication.

Preview: The Power of Interaction. This comparison of the interactive `MATMULT` protocol to Freivalds’ non-interactive one exemplifies the power of interaction in verification. Interaction buys the verifier the ability to ensure that the prover correctly materialized intermediate values in a computation (in this case, the entries of the product matrix C), without requiring the prover to explicitly materialize those values to the verifier. This point will become clearer later, when we cover the counting triangles protocol in Section 4.5.1. Roughly speaking, in that protocol, the prover convinces the verifier it correctly determined the squared adjacency matrix of the input graph, without ever materializing the squared adjacency matrix to the verifier.

Preview: Other Protocols for `MATMULT`. An alternate interactive `MATMULT` protocol can be obtained by applying the GKR protocol (covered later in Section 4.6) to a circuit \mathcal{C} that computes the product C of two input matrices A, B . The verifier in this protocol runs in $O(n^2)$ time, and the prover runs in time $O(S)$, where S is the number of gates in \mathcal{C} .

The advantage of the `MATMULT` protocol described in this section is two-fold. First, it does not care how the prover finds the right answer. In contrast, the GKR protocol demands that the prover compute the answer matrix C in a prescribed manner, namely by evaluating the circuit \mathcal{C} gate-by-gate. Second, the prover in the protocol of this section simply finds the right answer and then does $O(n^2)$ extra work to prove correctness. This $O(n^2)$ term is a low-order additive overhead, assuming that there is no linear-time algorithm for matrix multiplication. In contrast, the GKR protocol introduces at least a constant factor overhead for the prover. In practice, this is the difference between a prover that runs many times slower than an (unverifiable) `MATMULT` algorithm, and a prover that runs a fraction of a percent slower [Tha13].

4.4.2 The Protocol

Given $n \times n$ input matrix A, B , recall that we denote the product matrix $A \cdot B$ by C . And as in Section 4.3, we interpret A, B , and C as functions f_A, f_B, f_C mapping $\{0, 1\}^{\log n} \times \{0, 1\}^{\log n}$ to \mathbb{F} via:

$$f_A(i_1, \dots, i_{\log n}, j_1, \dots, j_{\log n}) = A_{ij}.$$

As usual, \tilde{f}_A, \tilde{f}_B , and \tilde{f}_C denote the MLEs of f_A, f_B , and f_C .

It is cleanest to describe the protocol for MATMULT as a protocol for evaluating \tilde{f}_C at any given point $(r_1, r_2) \in \mathbb{F}^{\log n \times \log n}$. As we explain later (see Section 4.5), this turns out to be sufficient for application problems such as graph diameter and triangle counting.

The protocol for computing $\tilde{f}_C(r_1, r_2)$ exploits the following explicit representation of the polynomial $\tilde{f}_C(x, y)$.

Lemma 4.2. $\tilde{f}_C(x, y) = \sum_{b \in \{0, 1\}^{\log n}} \tilde{f}_A(x, b) \cdot \tilde{f}_B(b, y)$. Here, the equality holds as formal polynomials in the coordinates of x and y .

Proof. The left and right hand sides of the equation appearing in the lemma statement are both multilinear polynomials in the coordinates of x and y . Since the MLE of C is unique, we need only check that the left and right hand sides of the equation agree for all *Boolean* vectors $i, j \in \{0, 1\}^{\log n}$. That is, we must check that for Boolean vectors $i, j \in \{0, 1\}^{\log n}$,

$$f_C(i, j) = \sum_{k \in \{0, 1\}^{\log n}} f_A(i, k) \cdot f_B(k, j). \quad (4.8)$$

But this is immediate from the definition of matrix multiplication. □

With Lemma 4.2 in hand, the interactive protocol is immediate: we compute $\tilde{f}_C(r_1, r_2)$ by applying the sum-check protocol to the $(\log n)$ -variate polynomial $g(z) := \tilde{f}_A(r_1, z) \cdot \tilde{f}_B(z, r_2)$.

Example. Consider the 2×2 matrices $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ over \mathbb{F}_5 . One can check that

$$A \cdot B = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}.$$

Viewing A and B as a functions mapping $\{0, 1\}^2 \rightarrow \mathbb{F}_5$,

$$\tilde{f}_A(x_1, x_2) = (1 - x_1)x_2 + 2x_1(1 - x_2) = -3x_1x_2 + 2x_1 + x_2,$$

and

$$\tilde{f}_B(x_1, x_2) = (1 - x_1)(1 - x_2) + 4x_1x_2 = 5x_1x_2 - x_1 - x_2 + 1 = 1 - x_1 - x_2,$$

where the final equality used the fact that we are working over \mathbb{F}_5 , so the coefficient 5 is the same as the coefficient 0.

Observe that

$$\begin{aligned} \sum_{b \in \{0, 1\}} \tilde{f}_A(x_1, b) \cdot \tilde{f}_B(b, x_2) &= \tilde{f}_A(x_1, 0) \cdot \tilde{f}_B(0, x_2) + \tilde{f}_A(x_1, 1) \cdot \tilde{f}_B(1, x_2) \\ &= 2x_1 \cdot (1 - x_2) + (-x_1 + 1) \cdot (-x_2) = -x_1x_2 + 2x_1 - x_2. \end{aligned} \quad (4.9)$$

Meanwhile, viewing C as a function f_C mapping $\{0, 1\}^2 \rightarrow \mathbb{F}_5$, we can calculate via Lagrange Interpolation:

$$\tilde{f}_C(x_1, x_2) = 4(1 - x_1)x_2 + 2x_1(1 - x_2) = -6x_1x_2 + 2x_1 + 4x_2 = -x_1x_2 + 2x_1 - x_2,$$

where the final equality uses that $6 \equiv 1$ and $4 \equiv -1$ when working modulo 5. Hence, we have verified that Lemma 4.2 indeed holds for this particular example.

4.4.3 Discussion of costs.

Rounds and communication cost. Since g is a $(\log n)$ -variate polynomial of degree 2 in each variable, the total communication is $O(\log n)$ field elements, spread over $\log n$ rounds.

\mathcal{V} 's runtime. At the end of the sum-check protocol, \mathcal{V} must evaluate $g(r_3) = \tilde{f}_A(r_1, r_3) \cdot \tilde{f}_B(r_3, r_2)$. To perform this evaluation, it suffices for \mathcal{V} to evaluate $\tilde{f}_A(r_1, r_3)$ and $\tilde{f}_B(r_3, r_2)$. Since \mathcal{V} is given the matrices A and B as input, Lemma 3.8 implies that both evaluations can be performed in $O(n^2)$ time.

\mathcal{P} 's runtime. Recall that in each round k of the sum-check protocol \mathcal{P} sends a quadratic polynomial $g_k(X_k)$ claimed to equal:

$$\sum_{b_{k+1} \in \{0,1\}} \dots \sum_{b_{\log n} \in \{0,1\}} g(r_{3,1}, \dots, r_{3,k-1}, X_i, b_{k+1}, \dots, b_{\log n}),$$

and to specify $g_k(X_k)$, \mathcal{P} can just send the values $g_i(0)$, $g_i(1)$, and $g_i(2)$. Thus, it is enough for \mathcal{P} to evaluate g at all points of the form

$$(r_{3,1}, \dots, r_{3,k-1}, \{0, 1, 2\}, b_{k+1}, \dots, b_{\log n}) : (b_{k+1}, \dots, b_{\log n}) \in \{0, 1\}^{\log n - k}. \quad (4.10)$$

There are $3 \cdot n/2^k$ such points in round k .

We describe three separate methods to perform these evaluations. The first method is the least sophisticated and requires $\Theta(n^3)$ total time. The second method reduces the runtime to $\Theta(n^2)$ per round, for a total runtime bound of $\Theta(n^2 \log n)$ over all $\log n$ rounds. The third method is more sophisticated still—it enables the prover to *reuse work* across rounds, ensuring that \mathcal{P} 's runtime in round k is bounded by $O(n^2/2^k)$. Hence, the prover's total runtime is $O(\sum_k n^2/2^k) = O(n^2)$.

Method 1. As described when bounding \mathcal{V} 's runtime, g can be evaluated at any point in $O(n^2)$ time. Since there are $3 \cdot n/2^k$ points at which \mathcal{P} must evaluate g in round k , this leads to a total runtime for \mathcal{P} of $O(\sum_k n^3/2^k) = O(n^3)$.

Method 2. To improve on the $O(n^3)$ runtime of Method 1, the key is to exploit the fact that $3 \cdot n/2^k$ points at which \mathcal{P} needs to evaluate g in round k are not arbitrary points in $\mathbb{F}^{\log n}$, but are instead highly structured. Specifically, each such point z is in the form of Equation (4.10), and hence the trailing coordinates of z are all Boolean (i.e., $\{0, 1\}$ -valued). As explained below, this property ensures that **each entry A_{ij} of A contributes to $g(r_{3,1}, \dots, r_{3,k-1}, \{0, 1, 2\}, b_{k+1}, \dots, b_{\log n})$ for only one tuple $(b_{k+1}, \dots, b_{\log n}) \in \{0, 1\}^{\log n - k}$, and similarly for each entry of B_{ij}** . Hence, \mathcal{P} can make a single pass over the matrices A and B , and for each entry A_{ij} or B_{ij} , \mathcal{P} only needs to update $g(z)$ for the three relevant tuples z of the form $(r_{3,1}, \dots, r_{3,k-1}, \{0, 1, 2\}, b_{k+1}, \dots, b_{\log n})$.

In more detail, in order to evaluate g at any input z , it suffices for \mathcal{P} to evaluate $\tilde{f}_A(r_1, z)$ and $\tilde{f}_B(z, r_2)$. We'll explain the case of evaluating $\tilde{f}_A(r_1, z)$ at all relevant points z , since the case of $\tilde{f}_B(z, r_2)$ is identical. From Lemma 3.6 (Lagrange Interpolation), $\tilde{f}_A(r_1, z) = \sum_{i,j \in \{0,1\}^{\log n}} A_{ij} \chi_{(i,j)}(r_1, z)$. For any input z of the form $(r_{3,1}, \dots, r_{3,k-1}, \{0, 1, 2\}, b_{k+1}, \dots, b_{\log n})$, notice that $\chi_{i,j}(r_1, z) = 0$ unless $(j_{k+1}, \dots, j_{\log n}) = (b_{k+1}, \dots, b_{\log n})$. This is because, for any coordinate ℓ such that $j_\ell \neq b_\ell$, the factor $(j_\ell b_\ell + (1 - j_\ell)(1 - b_\ell))$ appearing in the product defining $\chi_{(i,j)}$ equals 0 (see Equation (3.1)).

This enables \mathcal{P} to evaluate $\tilde{f}_A(r_1, z)$ in round k at all points z of the form of Equation (4.10) with a single pass over A : when \mathcal{P} encounters entry A_{ij} of A , \mathcal{P} updates $\tilde{f}_A(z) \leftarrow \tilde{f}_A(z) + \chi_{i,j}(z)$ for the three relevant values of z .

Method 3. To shave the last factor of $\log n$ off \mathcal{P} 's runtime, the idea is to have \mathcal{P} reuse work across rounds. Very roughly speaking, the key fact that enables this is the following:

Informal Fact. If two entries $(i, j), (i', j') \in \{0, 1\}^{\log n} \times \{0, 1\}^{\log n}$ agree in their last ℓ bits, then $A_{i,j}$ and $A_{i',j'}$ contribute to the same three points in each of the final ℓ rounds of the protocol.

The specific points that they contribute to in each round $k \geq \log(n) - \ell$ are the ones of the form

$$z = (r_{3,1}, \dots, r_{3,k-1}, \{0, 1, 2\}, b_{k+1}, \dots, b_{\log n}),$$

where $b_{k+1} \dots b_{\log n}$ equal the trailing bits of (i, j) and (i', j') . This turns out to ensure that \mathcal{P} can treat (i, j) and (i', j') as a single entity thereafter. There are only $O(n^2/2^k)$ entities of interest after k variables have been bound (out of the $2 \log n$ variables over which \tilde{f}_A is defined). So the total work that \mathcal{P} invests over the course of the protocol is

$$O\left(\sum_{k=1}^{2 \log n} n^2/2^k\right) = O(n^2).$$

In more detail, the **Informal Fact** stated above is captured by the proof of the following lemma.

Lemma 4.3. *Suppose that p is an ℓ -variate multilinear polynomial over field \mathbb{F} and that A is an array of length 2^ℓ such that for each $x \in \{0, 1\}^\ell$, $A[x] = p(x)$.⁴⁰ Then for any $r_1 \in \mathbb{F}$, there is an algorithm running in time $O(2^\ell)$ that, given r_1 and A as input, computes an array B of length $2^{\ell-1}$ such that for each $x' \in \{0, 1\}^{\ell-1}$, $B[x'] = p(r_1, x')$.*

Proof. The proof is reminiscent of that of Lemma 3.8. Specifically, we can express the multilinear polynomial

$$p(x_1, x_2, \dots, x_\ell) = x_1 \cdot p(1, x_2, \dots, x_\ell) + (1 - x_1) \cdot p(0, x_2, \dots, x_\ell). \quad (4.11)$$

Indeed, the right hand side is clearly a multilinear polynomial that agrees with p at all inputs in $\{0, 1\}^\ell$, and hence must equal p by Fact 3.5. The algorithm to compute B iterates over every value $x' \in \{0, 1\}^{\ell-1}$ and sets $B[x'] \leftarrow r_1 \cdot A[1, x'] + (1 - r_1) \cdot A[0, x']$.⁴¹ \square

Lemma 4.3 captures Informal Fact because, while inputs $(0, x')$ and $(1, x')$ to p both contribute to $B[x']$, they contribute to no other entries of the array B . As we will see when we apply repeatedly apply Lemma 4.3 to compute the prover's messages in the sum-check protocol, once $B[x']$ is computed, the prover only needs to know $B[x']$, not $p(0, x')$ or $p(1, x')$ individually.

Lemma 4.4. *Let h be any ℓ -variate multilinear polynomial over field \mathbb{F} for which all evaluations of $h(x)$ for $x \in \{0, 1\}^\ell$ can be computed in time $O(2^\ell)$. Let $r_1, \dots, r_\ell \in \mathbb{F}$ be any sequence of ℓ field elements. Then there is an algorithm that runs in time $O(2^\ell)$ and computes the following quantities:*

$$\{h(r_1, \dots, r_{i-1}, \{0, 1, 2\}, b_{i+1}, \dots, b_\ell)\}_{i=1, \dots, \ell; b_{i+1}, \dots, b_\ell \in \{0, 1\}} \quad (4.12)$$

⁴⁰Here, we associate bit-vectors x of length ℓ with indices into the array A of length 2^ℓ in the natural way.

⁴¹As in the statement of the lemma, here we associate bit-vectors x of length ℓ with indices into the array A of length 2^ℓ in the natural way, and similarly bit-vectors x' of length $\ell - 1$ with indices into the array B of length $2^{\ell - 1}$.

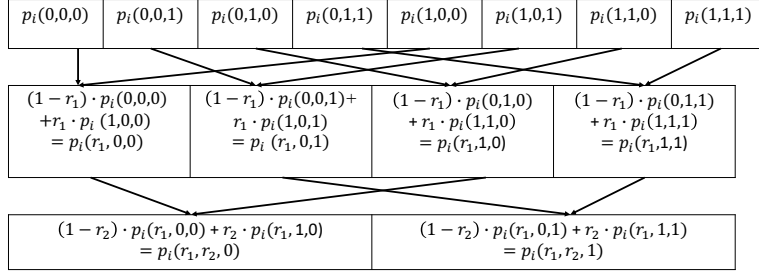


Figure 4.4: Depiction of the round-by-round evolution of the honest prover’s internal data structure devoted to the polynomial p_i in Lemma 4.5 in the case $\ell = 3$ (recall this lemma considers the sum-check protocol applied to compute $\sum_{x \in \{0,1\}^\ell} p_1(x) \cdots p_k(x)$ when each p_i is multilinear). The top row is used by the prover to compute its prescribed message in the first round, the middle row for the second round, and the bottom row for the third round.

Proof. Let

$$S_i = \{h(r_1, \dots, r_{i-1}, b_i, b_{i+1}, \dots, b_\ell)\}_{b_i, \dots, b_\ell \in \{0,1\}}.$$

Given all values in S_i , applying Lemma 4.3 to the $(\ell - i + 1)$ -variate multilinear polynomial $p(X_i, \dots, X_\ell) = h(r_1, \dots, r_{i-1}, X_i, \dots, X_\ell)$ implies that all values in S_{i+1} can be computed in time $O(2^{\ell-i})$.

Equation (4.11) further implies

$$h(r_1, \dots, r_{i-1}, 2, b_{i+1}, \dots, b_\ell) = 2 \cdot h(r_1, \dots, r_{i-1}, 1, b_{i+1}, \dots, b_\ell) - h(r_1, \dots, r_{i-1}, 0, b_{i+1}, \dots, b_\ell),$$

and hence the values

$$\{h(r_1, \dots, r_{i-1}, 2, b_{i+1}, \dots, b_\ell)\}_{b_i, \dots, b_\ell \in \{0,1\}}$$

can also be computed in $O(2^{\ell-i})$ time given the values in S_i .

It follows the total time required to compute all values in Equation (4.12) is $O(\sum_{i=1}^\ell 2^{\ell-i}) = O(2^\ell)$. \square

Lemma 4.5. (Implicit in [CTY11, Appendix B], see also [Tha13, XZZ⁺19]) *Let p_1, p_2, \dots, p_k be ℓ -variate multilinear polynomials. Suppose that for each p_i there is an algorithm that evaluates p_i at all inputs in $\{0, 1\}^\ell$ in time $O(2^\ell)$. Let $g = p_1 \cdot p_2 \cdots p_k$ be the product of these multilinear polynomials. Then when the sum-check protocol is applied to the polynomial g , the honest prover can be implemented in $O(k \cdot 2^\ell)$ time.*

Proof. As explained in Equation (4.10), the dominant cost in the honest prover’s computation in the sum-check protocol lies in evaluating g at points of the form referred to in Lemma 4.4 (see Equation (4.12)). To obtain these evaluations, it clearly suffices to evaluate p_1, \dots, p_k at each one of these points, and multiply the results in time $O(k)$ per point. Lemma 4.4 guarantees that each p_i can be evaluated at the relevant points in $O(2^\ell)$ time, yielding a total runtime of $O(k \cdot 2^\ell)$. See Figure 4.4 for a depiction of the honest prover’s computation in the case $\ell = 3$. \square

In the matrix multiplication protocol of this section, the sum-check protocol is applied to the $(\log_2 n)$ -variate polynomial $g(X_3) = \tilde{f}_A(r_1, X_3) \cdot \tilde{f}_B(X_3, r_2)$. The multilinear polynomial $\tilde{f}_A(r_1, X_3)$ can be evaluated at all inputs in $\{0, 1\}^{\log n}$ in $O(n^2)$ time, by applying Lemma 4.4 with $h = \tilde{f}_A$, and observing that the necessary evaluations of $\tilde{f}_A(r_1, X_3)$ are a subset of the points in Equation (4.12) (with $i = \log n$, $(r_1, \dots, r_{\log n}) = r_1$). Similarly, $\tilde{f}_B(X_3, r_2)$ can be evaluated at all inputs in $\{0, 1\}^{\log n}$ in $O(n^2)$ time. Given all of these evaluations, Lemma 4.5 implies that the prover can execute its part of the sum-check protocol in just $O(n)$ additional time.

This completes the explanation of how the prover in the matrix multiplication protocol of this section executes its part of the sum-check protocol in $O(n^2)$ total time.

Communication	Rounds	\mathcal{V} time	\mathcal{P} time
$O(\log n)$ field elements	$\log n$	$O(n^2)$	$T + O(n^2)$

Table 4.3: Costs of the MATMULT protocol of Section 4.4 when applied to $n \times n$ matrices A and B . Here, T is the time required by \mathcal{P} to compute the product matrix $C = A \cdot B$.

4.5 Applications of the Super-Efficient MATMULT IP

Why does an IP for computing $\tilde{f}_C(r_1, r_2)$ rather than the full product matrix $C = A \cdot B$ suffice in applications? This section answers this question via several examples. With the exception of Section 4.5.4, all of the protocols in this section enable the honest prover to run the best-known algorithm to solve the problem at hand, and then do a low-order amount of extra work to prove the answer is correct. We refer to such IPs as *super-efficient* for the prover. There are no other known IPs or argument systems that achieve this super-efficiency while keeping the proof length sublinear in the input size.

4.5.1 A Super-Efficient IP For Counting Triangles

Algorithms often invoke MATMULT to generate crucial *intermediate values* compute some product matrix C , but are not interested in the product matrix itself. For example, the fastest known algorithm for counting triangles in dense graphs works as follows. If A is the adjacency matrix of a simple graph, the algorithm first computes A^2 (it is known how to accomplish this in time $O(n^{2.3728639})$ [LG14]), and then outputs (1/6 times)

$$\sum_{i,j \in \{1, \dots, n\}} (A^2)_{ij} \cdot A_{ij}. \quad (4.13)$$

It is not hard to see that Equation (4.13) quantity is six times the number of triangles in the graph, since $(A^2)_{i,j}$ counts the number of common neighbors of vertices i and j , and hence $(A^2)_{i,j} \cdot A_{ij}$ equals the number of vertices k such that (i, j) , (j, k) and (k, i) are all edges in the graph.⁴²

Clearly, the matrix A^2 is not of intrinsic interest here, but rather is a useful intermediate object from which the final answer can be quickly derived. As we explain in this section, it is possible to give an IP for counting triangles in which \mathcal{P} essentially establishes that he correctly materialized A^2 and used it to generate the output via Equation (4.13). Crucially, \mathcal{P} will accomplish this with only logarithmic communication (i.e., without sending A^2 to the verifier), and while doing very little extra work beyond determining A^2 .

The Protocol. As in Section 4.3, let \mathbb{F} be a finite field of size $p \geq 6n^3$, where p is a prime, and let us view all entries of A as elements of \mathbb{F} . Define the functions $f_A(x, y), f_{A^2}(x, y): \{0, 1\}^{\log n} \times \{0, 1\}^{\log n} \rightarrow \mathbb{F}$ that interprets x and y as the binary representations of some integers i and j between 1 and n , and outputs $A_{i,j}$ and $(A^2)_{i,j}$ respectively. Let \tilde{f}_A and \tilde{f}_{A^2} denote the multilinear extensions of f_A and f_{A^2} over \mathbb{F} .

Then the expression in Equation (4.13) equals $\sum_{x,y \in \{0,1\}^{\log n}} \tilde{f}_{A^2}(x, y) \cdot \tilde{f}_A(x, y)$. This quantity can be computed by applying the sum-check protocol to the multi-quadratic polynomial $\tilde{f}_{A^2} \cdot \tilde{f}_A$. At the end of this protocol, the verifier needs to evaluate $\tilde{f}_{A^2}(r_1, r_2) \cdot \tilde{f}_A(r_1, r_2)$ for a randomly chosen input $(r_1, r_2) \in \mathbb{F}^{\log n} \times \mathbb{F}^{\log n}$. The verifier can evaluate $\tilde{f}_A(r_1, r_2)$ unaided in $O(n^2)$ time using Lemma 3.8. While the verifier cannot evaluate $\tilde{f}_{A^2}(r_1, r_2)$ without computing the matrix A^2 (which is as hard as solving the counting triangles problem

⁴²Eq. (4.13) counts each triangle six times: for each unordered triple of distinct nodes (a, b, c) forming a triangle in the graph, the triangle gets counted in Eq. (4.13) when (i, j) is set to each of the ordered pairs (a, b) , (b, a) , (b, c) , (c, b) , (a, c) , and (c, a) .

on her own), evaluating $\tilde{f}_{A^2}(r_1, r_2)$ is exactly the problem that the MatMult IP of Section 4.4.2 was designed to solve (as $A^2 = A \cdot A$), so we simply invoke that protocol to compute $\tilde{f}_{A^2}(r_1, r_2)$.

Example. Consider the example from Section 4.3, in which the input matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case,

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

One can check that

$$\tilde{f}_A(X, Y) = X \cdot (1 - Y) + Y \cdot (1 - X),$$

and

$$\tilde{f}_{A^2}(X, Y) = X \cdot Y + (1 - Y) \cdot (1 - X).$$

The counting triangles protocol in this section first applies the sum-check protocol to the following bivariate polynomial that has degree 2 in both of its variables:

$$\tilde{f}_{A^2}(X, Y) \cdot \tilde{f}_A(X, Y) = (X \cdot (1 - Y) + Y \cdot (1 - X)) \cdot (X \cdot Y + (1 - X) \cdot (1 - Y)).$$

It is easy to check that this polynomial evaluates to 0 for all four inputs in $\{0, 1\}^2$, so applying the sum-check protocol to this polynomial reveals to the verifier that $\sum_{(x,y) \in \{0,1\}^2} \tilde{f}_{A^2}(x,y) \cdot \tilde{f}_A(x,y) = 0$.

At the end of the sum-check protocol applied to this polynomial, the verifier needs to evaluate \tilde{f}_{A^2} and \tilde{f}_A at a randomly chosen input $(r_1, r_2) \in \mathbb{F} \times \mathbb{F}$. The verifier evaluates $\tilde{f}_A(r_1, r_2)$ on its own. To compute $\tilde{f}_{A^2}(r_1, r_2)$, the matrix multiplication IP is invoked. This protocol applies the sum-check protocol a second time, to the univariate quadratic polynomial

$$s(X) := \tilde{f}_A(r_1, X) \cdot \tilde{f}_A(X, r_2) = (r_1(1 - X) + (1 - r_1)X) \cdot (X(1 - r_2) + r_2(1 - X)).$$

This reveals to the verifier that

$$\tilde{f}_{A^2}(r_1, r_2) = s(0) + s(1) = r_1 r_2 + (1 - r_1)(1 - r_2).$$

At the end of this second invocation of the sum-check protocol, the verifier needs to evaluate $s(r_3)$ for a randomly chosen $r_3 \in \mathbb{F}$. To do this, it suffices to evaluate $\tilde{f}_A(r_1, r_3)$ and $\tilde{f}_A(r_3, r_2)$, both of which the verifier computes on its own.

Costs of the Counting Triangles Protocol. In a sentence, the number of rounds, communication size, and verifier runtime of the IP of this section are all identical to the counting triangles protocol we saw earlier in Section 4.3 (namely, $O(\log n)$ rounds and communication, and $O(n^2)$ time verifier). The big advantage of the protocol of this section is in prover time: the prover in this section merely has to compute the matrix A^2 (it does not matter how \mathcal{P} chooses to compute A^2), and then does $O(n^2)$ extra work to compute the prescribed messages in the two invocations of the sum-check protocol. Up to the additive $O(n^2)$ term, this matches the amount of work performed by the fastest known (unverifiable) algorithm for counting triangles. The additive $O(n^2)$ will likely be a low-order cost for \mathcal{P} , since computing A^2 alone will require time at least n^ω , where ω is the matrix multiplication constant (it is widely believed that ω is strictly greater than 2).

Communication and Rounds. In more detail, the application of sum-check to the polynomial $\tilde{f}_{A^2} \cdot \tilde{f}_A$ requires $2 \log n$ rounds, with 3 field elements sent from prover to verifier in each round. The matrix multiplication IP

used to compute $\tilde{f}_{A^2}(r_1, r_2)$ requires an additional $\log n$ rounds, with 3 field elements sent from the prover to verifier in each round. This means there are $3 \log n$ rounds in total, with $9 \log n$ field elements sent from the prover to the verifier (and $3 \log n$ sent from the verifier to the prover). This round complexity and communication cost is identical to the counting triangles protocol from Section 4.3.

Verifier runtime. The verifier is easily seen to run in $O(n^2)$ time in total—it’s runtime is dominated by the cost of evaluating \tilde{f}_A at three random inputs: (r_1, r_2) , (r_2, r_3) , and (r_1, r_3) . This too is identical to the verifier cost in the counting triangles protocol from Section 4.3.

Prover runtime. Once the prover knows A^2 , the prover’s messages in both the sum-check applied to the polynomial $\tilde{f}_{A^2} \cdot \tilde{f}_A$, and in the matrix multiplication IP of Section 4.4.2, can be derived in $O(n^2)$ time. Specifically, Method 3 of Section 4.4.3 achieves an $O(n^2)$ time prover in the matrix multiplication IP, and the same techniques show that, if \mathcal{P} knows all of the entries of the matrix A^2 , then in $O(n^2)$ time \mathcal{P} can compute the prescribed messages when applying the sum-check protocol to the polynomial $\tilde{f}_{A^2} \cdot \tilde{f}_A$.

4.5.2 A Super-Efficient IP for Matrix Powers

Let A be an $n \times n$ matrix with entries from field \mathbb{F} , and suppose a verifier wants to evaluate a single entry of the powered matrix A^k for a large integer k (for concreteness, let’s say \mathcal{V} is interested in learning entry $(A^k)_{n,n}$, and k and n are powers of 2). As we now explain, the MatMult IP of Section 4.4 gives a way to do this, with $O(\log(k) \cdot \log(n))$ rounds and communication, and a verifier that runs in $O(n^2 + \log(k) \log(n))$ time.

Clearly we can express the matrix A^k as a product of smaller powers of A :

$$A^k = A^{k/2} \cdot A^{k/2}. \quad (4.14)$$

Hence, letting g_ℓ denote the multilinear extension of the matrix A^ℓ , we can try to exploit Equation (4.14) by applying the MatMult IP to compute $(A^k)_{n,n} = g_k(\mathbf{1}, \mathbf{1})$.

But at the end of this MatMult IP, the verifier needs to evaluate $g_{k/2}$ at two points. The verifier cannot do this since she doesn’t know $A^{k/2}$.

Reducing two points to one. There are known interactive proof techniques that enable a verifier to reduce evaluating a polynomial $g_{k/2}$ at the two points to evaluating $g_{k/2}$ at a single point. The same issue comes up in the GKR protocol covered in Section 4.6. We defer the details of this technique to that section (see the paragraph entitled “Reducing to Verification of a Single Point” in Section 4.6.4).

Recursion to the Rescue. After reducing two points to one, the verifier is left with the task of evaluating $g_{k/2}$ at a single input, say $(r_1, r_2) \in \mathbb{F}^{\log n} \times \mathbb{F}^{\log n}$. Since $g_{k/2}$ is the multilinear extension of the matrix $A^{k/2}$ (viewed in the natural way as a function $f_{A^{k/2}}$ mapping $\{0, 1\}^{\log n} \times \{0, 1\}^{\log n} \rightarrow \mathbb{F}$), and $A^{k/2}$ can be decomposed as $A^{k/4} \cdot A^{k/4}$, the verifier can recursively apply the MatMult protocol to compute $g_{k/2}(r_1, r_2)$. This runs into the same issues as before, namely that to run the MatMult protocol, the verifier needs to evaluate $g_{k/4}$ at two points, which can in turn be reduced to the task of evaluating $g_{k/4}$ at a single point. This can again be handled recursively as above. After $\log k$ layers of recursion, there is no need to recurse further since the verifier can evaluate $g_1 = \tilde{f}_A$ at any desired input in $O(n^2)$ time using Lemma 3.8. That is, Lemma 3.8 states that the multilinear extension of the input matrix A can be evaluated at any desired point in linear time.

4.5.3 A General Paradigm for IPs with Super-Efficient Provers

Beyond algorithms for counting triangles, there are other algorithms that invoke MATMULT to compute some product matrix C , and then apply some post-processing to C to compute an answer that is much smaller than C itself (often the answer is just a single number, rather than an $n \times n$ matrix). In these settings, \mathcal{V} can apply a general-purpose protocol, such as the GKR protocol that will be presented in Section 4.6, to verify that the post-processing step was correctly applied to the product matrix C . As we will see in Section 4.6, at the end of the application of the GKR protocol, \mathcal{V} needs to evaluate $\tilde{f}_C(r_1, r_2)$ at a randomly chosen point $(r_1, r_2) \in \mathbb{F}^{\log n \times \log n}$. \mathcal{V} can do this using the MATMULT protocol described above.

Crucially, this post-processing step typically requires time linear in the size of C . So \mathcal{P} 's runtime in this application of the GKR protocol will be proportional to the size of (a circuit computing) the post-processing step, which is typically just $\tilde{O}(n^2)$.

As a concrete example, consider the problem of computing the diameter of a directed graph G . Let A denote the adjacency matrix of G , and let I denote the $n \times n$ identity matrix. Then the diameter of G is the least positive number d such that $(A + I)_{ij}^d \neq 0$ for all (i, j) . This yields the following natural protocol for diameter. \mathcal{P} sends the claimed output d to \mathcal{V} , as well as an (i, j) such that $(A + I)_{ij}^{d-1} = 0$. To confirm that d is the diameter of G , it suffices for \mathcal{V} to check two things: first, that all entries of $(A + I)^d$ are nonzero, and second that $(A + I)_{ij}^{d-1}$ is indeed zero.

The first task is accomplished by combining the MATMULT protocol with the GKR protocol as follows. Let d_j denote the j th bit in the binary representation of d . Then $(A + I)^d = \prod_j^{[\log d]} (A + I)^{d_j 2^j}$, so computing the number of nonzero entries of $D_1 = (A + I)^d$ can be computed via a sequence of $O(\log d)$ matrix multiplications, followed by a post-processing step that computes the number of nonzero entries of D_1 . We can apply the GKR protocol to verify this post-processing step, but at the end of the protocol, \mathcal{V} needs to evaluate the multilinear extension of D_1 at a random point (as usual, when we refer to the multilinear extension of D_1 , we are viewing D_1 as a function mapping $\{0, 1\}^{\log n} \times \{0, 1\}^{\log n} \rightarrow \mathbb{F}$ in the natural way). \mathcal{V} cannot do this without help, so \mathcal{V} outsources even this computation to \mathcal{P} , by using $O(\log d)$ invocations of the MATMULT protocol described above.

The second task, of verifying that $(A + I)_{ij}^{d-1} = 0$, is similarly accomplished using $O(\log d)$ invocations of the MATMULT protocol—since \mathcal{V} is only interested in one entry of $(A + I)^{d-1}$, \mathcal{P} need not send the matrix $(A + I)^{d-1}$ in full, and the total communication here is just $\text{polylog}(n)$.

Ultimately, \mathcal{V} 's runtime in this diameter protocol is $O(m \log n)$, where m is the number of edges in G . \mathcal{P} 's runtime in the above diameter protocol matches the best known unverifiable diameter algorithm up to a low-order additive term [Sei95, Yus10], and the communication is just $\text{polylog}(n)$.

4.5.4 An IP for Small-Space Computations (and IP = PSPACE)

In this section, we use the matrix-powering protocol to re-prove the following important result of Goldwasser, Kalai, and Rothblum (GKR) [GKR08]: all problems solvable in logarithmic space have an IP with a quasilinear-time verifier, polynomial time prover, and polylogarithmic proof length.

The basic idea of the proof is that executing any Turing Machine M that uses s bits of space can be reduced to the problem of computing a single entry of A^{2^s} for a certain matrix A (A is in fact the configuration graph of M). So one can just apply the matrix-powering IP to A to determine the output of M . While A is a huge matrix (it has at least 2^s rows and columns), configuration graphs have a ton of structure, and this enables the verifier to evaluate \tilde{f}_A at a single input in $O(s \cdot n)$ time. If s is logarithmic in the input size, then this means that the verifier in the IP runs in $O(n \log n)$ time.

The original paper of GKR proved the same result by constructing an arithmetic circuit for computing A^{2^s} and then applying a sophisticated IP for arithmetic circuit evaluation to that circuit (we cover this IP in Section 4.6 and the arithmetic circuit for computing A^{2^s} in Section 5.4). The approach described in this section is simpler, in that it directly applies a simple IP for matrix-powering, rather than the more complicated IP for the general circuit-evaluation problem.

Details. Let M be a Turing Machine that, when run on an m -bit input, uses at most s bits of space. Let $A(x)$ be the adjacency matrix of its *configuration graph* when M is run on input $x \in \{0, 1\}^m$. Here, the configuration graph has as its vertex set all of the possible states and memory configurations of the machine M , with a directed edge from vertex i to vertex j if running M for one step from configuration i on input x causes M to move to configuration j . Since M uses s bits of space, there are $O(2^s)$ many vertices of the configuration graph. This means that $A(x)$ is an $N \times N$ matrix for some $N = O(2^s)$. Note that if M never enters an infinite loop (i.e., never enters the same configuration twice), then M must trivially run in time at most N .

We can assume without loss of generality that M has a unique starting configuration and a unique accepting configuration; say for concreteness that these configurations correspond to vertices of the configuration graph with labels 1 and N . Then to determine whether M accepts input x , it is enough to determine if there is a length- N path from vertex 1 to vertex N in the configuration graph of M . This is equivalent to determining the $(1, N)$ 'th entry of the matrix $(A(x))^N$.

This quantity can be computed with the matrix power protocol of the previous section, which uses $O(s \cdot \log N)$ rounds and communication. At the end of the protocol, the verifier does need to evaluate the MLE of the matrix $A(x)$ at a randomly chosen input. This may seem like it should take up to $O(N^2)$ time, since A is a $N \times N$ matrix. However, the configuration matrix of any Turing Machine is highly structured, owing to the fact that at any time step, the machine only reads or writes to $O(1)$ memory cells, and only moves its read and write heads at most one cell to the left or right. This turns out to imply that the verifier can evaluate the MLE of A in $O(s \cdot m)$ time (we omit these details for brevity).

In total, the costs of the IP are as follows. The rounds and number of field elements communicated is $O(s \log N)$, the verifier's runtime is $O(s \log N + m \cdot s)$ and the prover's runtime is $\text{poly}(N)$. If $s = O(\log m)$, then these three costs are respectively $O(\log^2 m)$, $O(m \log m)$, and $\text{poly}(m)$. That is, the communication cost is polylogarithmic in the input size, the verifier's runtime is quasilinear, and the prover's runtime is polynomial.

Note that if $s = \text{poly}(m)$, then the verifier's runtime in this IP is $\text{poly}(m)$, recovering the famous result of LFKN [LFKN92] and Shamir [Sha92] that **IP = PSPACE**.

Additional Discussion. One disappointing feature of this IP is that, if the runtime of M is significantly less than $N \geq 2^s$, the prover will still take time at least N , because the prover has to explicitly generate powers of the configuration graph's adjacency matrix. This is particularly problematic if the space bound s is superlogarithmic in the input size m , since then 2^s is not even a polynomial in m . Effectively, the IP we just presented forces the prover to explore all possible configurations of M , even though when running M on input x , the machine will only enter a tiny fraction of such configurations. A breakthrough complexity-theory result of [RRR16] gave a very different IP that avoids this inefficiency for P (remarkably, their IP also requires only constantly many rounds of interaction).

4.6 The GKR Protocol and Its Efficient Implementation

4.6.1 Motivation

The goal of Section 4.2 was to develop an interactive proof for an intractable problem (such as #SAT [LFKN92] or TQBF [Sha92]), in which the verifier ran in polynomial time. The perspective taken in this section is different: it acknowledges that there are no “real world” entities that can act as the prover in the #SAT and TQBF protocols of earlier sections, since real world entities cannot solve large instances of PSPACE-complete or #P-complete problems in the worst case. We would really like a “scaled down” result, one that is useful for problems that can be solved in the real world, such as problems in the complexity classes P, or NC (capturing problems solvable by efficient parallel algorithms), or even L (capturing problems solvable in logarithmic space).

One may wonder what is the point of developing verification protocols for such easy problems. Can’t the verifier just ignore the prover and solve the problem without help? One answer is that this section will describe protocols in which the verifier runs much faster than would be possible without a prover. Specifically, \mathcal{V} will run linear time, doing little more than just reading the input.^{43,44}

Meanwhile, we will require that the prover not do much more than solve the problem of interest. Ideally, if the problem is solvable by a Random Access Machine or Turing Machine in time T and space s , we want the prover to run in time $O(T)$ and space $O(s)$, or as close to it as possible. At a minimum, \mathcal{P} should run in polynomial time.

Can the TQBF and #SAT protocols of prior sections be scaled down to yield protocols where the verifier runs in (quasi-)linear time for a “weak” complexity class like L? It turns out that it can, but the prover is not efficient.

Recall that in the #SAT protocol (as well as in the TQBF protocol of [Sha92]), \mathcal{V} ran in time $O(S)$, and \mathcal{P} ran in time $O(S^2 \cdot 2^N)$, when applied to a Boolean formula ϕ of size S over N variables. In principle, this yields an interactive proof for any problem solvable in space s : given an input $x \in \{0, 1\}^n$, \mathcal{V} first transforms x to an instance ϕ of TQBF (see, e.g., [AB09, Chapter 4] for a lucid exposition of this transformation, which is reminiscent of Savitch’s Theorem), and then applies the interactive proof for TQBF to ϕ .

However, the transformation yields a TQBF instance ϕ over $N = O(s \cdot \log T)$ variables when applied to a problem solvable in time T and space s . This results in a prover that runs in time in time $2^{O(s \cdot \log T)}$. This is superpolynomial (i.e., $n^{\Theta(\log n)}$), even if $s = O(\log n)$ and $T = \text{poly}(n)$. Until 2007, this was the state of the art in interactive proofs.

4.6.2 The GKR Protocol and Its Costs

Goldwasser, Kalai, and Rothblum [GKR08] described a remarkable interactive proof protocol that does achieve many of the goals set forth above. The protocol is best presented in terms of the (arithmetic) *circuit evaluation* problem. In this problem, \mathcal{V} and \mathcal{P} first agree on a (log-space uniform) arithmetic circuit \mathcal{C} of fan-in 2 over a finite field \mathbb{F} , and the goal is to compute the value of the output gate(s) of \mathcal{C} . A log-space uniform circuit \mathcal{C} is one that possesses a succinct implicit description, in the sense that there is a log-space algorithm

⁴³The protocols for counting triangles, matrix multiplication and powering, and graph diameter of Sections 4.3, 4.4, and 4.5 also achieved a linear-time verifier. But unlike the GKR protocol, those protocols were not general-purpose. As we will see, the GKR protocol is general-purpose in the sense that it solves the problem of arithmetic *circuit evaluation*, and any problem in P can be “efficiently” reduced to circuit evaluation (these reductions and the precise meaning of “efficiently” will be covered in Chapter 5).

⁴⁴Another answer is that such protocols can be combined with cryptography to render them zero-knowledge. And zero-knowledge protocols can be useful even when applied to “easy” problems. We defer discussion of zero-knowledge to later sections.

Communication	Rounds	\mathcal{V} time	\mathcal{P} time	Soundness error
$d \cdot \text{polylog}(S)$ field elements	$d \cdot \text{polylog}(S)$	$O(n + d \cdot \text{polylog}(S))$	$\text{poly}(S)$	$O(d \log(S)/ \mathbb{F})$

Table 4.4: Costs of the original GKR protocol [GKR08] when applied to any log-space uniform layered arithmetic circuit \mathcal{C} of size S and depth d over n variables defined over field \mathbb{F} . Section 4.6.5 describes methods from [CMT12] for reducing \mathcal{P} 's runtime to $O(S \log S)$, and reducing the $\text{polylog}(S)$ terms in the remaining costs to $O(\log S)$. It is now known how to achieve prover runtime of $O(S)$ for arbitrary layered arithmetic circuits \mathcal{C} (see Remark 4.5).

that takes as input the label of a gate a of \mathcal{C} , and is capable of determining all relevant information about that gate. That is, the algorithm can output the labels of all of a 's neighbors, and is capable of determining if a is an addition gate or a multiplication gate.

Letting S denote the size (i.e., number of gates) of \mathcal{C} and n the number of variables, the key feature of the GKR protocol is that the prover runs in time $\text{poly}(S)$ (we will see that \mathcal{P} 's time can even be made *linear* in S [CMT12, Tha13, XZZ⁺19]). If $S = 2^{o(n)}$, then this is much better than the #SAT protocol that we saw in an earlier section, where the prover required time exponential in the number of variables over which the #SAT instance was defined.

Moreover, the costs to the verifier in the GKR protocol grow linearly with the *depth* d of \mathcal{C} , and only logarithmically with S (in particular, the communication cost is $O(d \log S)$). Crucially, this means that \mathcal{V} can run in time *sublinear* in the size S of the circuit. At first glance, this might seem impossible—how can the verifier make sure the prover correctly evaluated \mathcal{C} if the verifier never even “looks” at all of \mathcal{C} ? The answer is that \mathcal{C} was assumed to have a succinct implicit description in the sense of being log-space uniform. This enables \mathcal{V} to “understand” the structure of \mathcal{C} without ever having to look at every gate individually.

Application: An IP for Parallel Algorithms. The complexity class **NC** consists of languages solvable by parallel algorithms in time $\text{polylog}(n)$ and total work $\text{poly}(n)$. Any problem in **NC** can be computed by a (log-space uniform) arithmetic circuit \mathcal{C} of polynomial size and polylogarithmic depth. Applying the GKR protocol to \mathcal{C} yields a polynomial time prover and a linear time verifier.

Application: Another IP for Space-Bounded Turing Machines. As discussed above, the TQBF protocol from earlier sections can be used to give a prover running in time $2^{O(\log s \cdot \log T)}$ and a verifier running in time $O(n \cdot \text{poly}(s))$ when applied to a language solvable in time T and space s . In particular, \mathcal{P} 's runtime is superpolynomial even if $s = O(\log n)$.

The IP for small-space computation in Section 4.5.4 provides a quantitative improvement on these costs. Specifically, the prover in that IP runs in time $\text{poly}(2^s)$, which is polynomial if $s = O(\log n)$. The verifier runs in time $O(n \cdot s)$. This is $O(n \log n)$ time if s is logarithmic in n .

In this section, we describe a different (but conceptually related) IP for small-space computation obtained via the GKR protocol. As we explain, any language \mathcal{L} solvable in time space s and time T is computed by an $O(\log s)$ -space uniform circuit \mathcal{C} of fan-in 2 and depth $d = O(s \cdot \log T)$ and size $S = \text{poly}(2^s)$. Roughly speaking, similar to the IP of Section 4.5.4, \mathcal{C} works by taking a Turing Machine or Random Access Machine M computing \mathcal{L} , and repeatedly squaring the adjacency matrix of M in order to determine whether there is a path from the start configuration of M to the accepting configuration of M (see Section 5.4 for details). Only $O(\log T)$ invocations of **MATMULT** are required to determine if a path of length T exists, and each invocation can be computed in depth $O(s)$, resulting in the $O(s \cdot \log T)$ bound on the depth of \mathcal{C} .

Applying the GKR protocol to \mathcal{C} , the prover runs in time $\text{poly}(S) = \text{poly}(2^s)$, which is polynomial if $s = O(\log n)$. The verifier runs in time $O(n + d \cdot \log S) = O(n + \text{poly}(s, \log T))$. This is $O(n)$ time if s is subpolynomial in n and T is subexponential in n .

Notice the above also provides an alternate proof that $\mathbf{PSPACE} \subseteq \mathbf{IP}$, as the verifier's runtime is $\text{poly}(n)$ as long as $s = \text{poly}(n)$.

4.6.3 Protocol Overview

As described above, \mathcal{P} and \mathcal{V} first agree on an arithmetic circuit \mathcal{C} of fan-in 2 over a finite field \mathbb{F} computing the function of interest. \mathcal{C} is assumed to be in layered form, meaning that the circuit can be decomposed into layers, and wires only connect gates in adjacent layers (if \mathcal{C} is not layered it can easily be transformed into a layered circuit \mathcal{C}' with a small blowup in size). Suppose that \mathcal{C} has depth d , and number the layers from 1 to d with layer d referring to the input layer, and layer 1 referring to the output layer.

In the first message, \mathcal{P} tells \mathcal{V} the (claimed) output(s) of the circuit. The protocol then works its way in iterations towards the input layer, with one iteration devoted to each layer. We will describe the gates in \mathcal{C} as having values: the value of an addition (respectively, multiplication) gate is set to be the sum (respectively, product) of its in-neighbors. The purpose of iteration i is to reduce a claim about the values of the gates at layer i to a claim about the values of the gates at layer $i + 1$, in the sense that it is safe for \mathcal{V} to assume that the first claim is true as long as the second claim is true. This reduction is accomplished by applying the sum-check protocol.

More concretely, the GKR protocol starts with a claim about the values of the output gates of the circuit, but \mathcal{V} cannot check this claim without evaluating the circuit herself, which is precisely what she wants to avoid. So the first iteration uses a sum-check protocol to reduce this claim about the outputs of the circuit to a claim about the gate values at layer 2 (more specifically, to a claim about an evaluation of the multilinear extension of the gate values at layer 2). Once again, \mathcal{V} cannot check this claim herself, so the second iteration uses another sum-check protocol to reduce the latter claim to a claim about the gate values at layer 3, and so on. Eventually, \mathcal{V} is left with a claim about the inputs to the circuit, and \mathcal{V} can check this claim without any help. This outline is depicted in Figures 4.5-4.8.

4.6.4 Protocol Details

Notation. Suppose we are given a layered arithmetic circuit \mathcal{C} of size S , depth d , and fan-in two (\mathcal{C} may have more than one output gate). Number the layers from 0 to d , with 0 being the output layer and d being the input layer. Let S_i denote the number of gates at layer i of the circuit \mathcal{C} . Assume S_i is a power of 2 and let $S_i = 2^{k_i}$. The GKR protocol makes use of several functions, each of which encodes certain information about the circuit.

Number the gates at layer i from 0 to $S_i - 1$, and let $W_i : \{0, 1\}^{k_i} \rightarrow \mathbb{F}$ denote the function that takes as input a binary gate label, and outputs the corresponding gate's value at layer i . As usual, let \tilde{W}_i denote the multilinear extension of W_i . See Figure 4.9, which depicts an example circuit \mathcal{C} and input to \mathcal{C} and describes the resulting function W_i for each layer i of \mathcal{C} .

The GKR protocol also makes use of the notion of a “wiring predicate” that encodes which pairs of wires from layer $i + 1$ are connected to a given gate at layer i in \mathcal{C} . Let $\text{in}_{1,i}, \text{in}_{2,i} : \{0, 1\}^{k_i} \rightarrow \{0, 1\}^{k_{i+1}}$ denote the functions that take as input the label a of a gate at layer i of \mathcal{C} , and respectively output the label of the first and second in-neighbor of gate a . So, for example, if gate a at layer i computes the sum of gates b and c at layer $i + 1$, then $\text{in}_{1,i}(a) = b$ and $\text{in}_{2,i}(a) = c$.

Define two functions, add_i and mult_i , mapping $\{0, 1\}^{k_i+2k_{i+1}}$ to $\{0, 1\}$, which together constitute the wiring predicate of layer i of \mathcal{C} . Specifically, these functions take as input three gate labels (a, b, c) , and return 1 iff $(b, c) = (\text{in}_{1,i}(a), \text{in}_{2,i}(a))$ and gate a is an addition (respectively, multiplication) gate. As usual, let $\tilde{\text{add}}_i$ and $\tilde{\text{mult}}_i$ denote the multilinear extensions of add_i and mult_i .

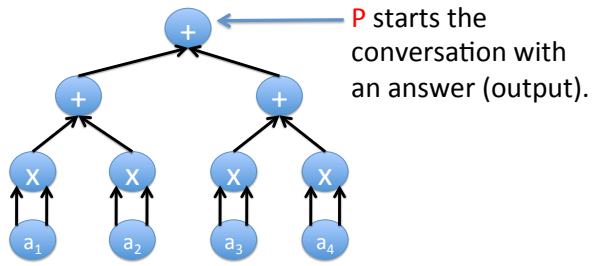


Figure 4.5: Start of GKR Protocol.

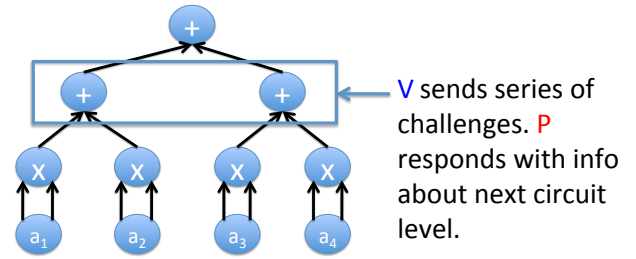


Figure 4.6: Iteration 1 reduces a claim about the output of \mathcal{C} to one about the MLE of the gate values in the previous layer.

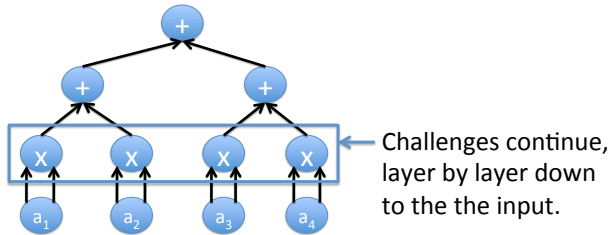


Figure 4.7: In general, iteration i reduces a claim about the MLE of gate values at layer i , to a claim about the MLE of gate values at layer $i + 1$.

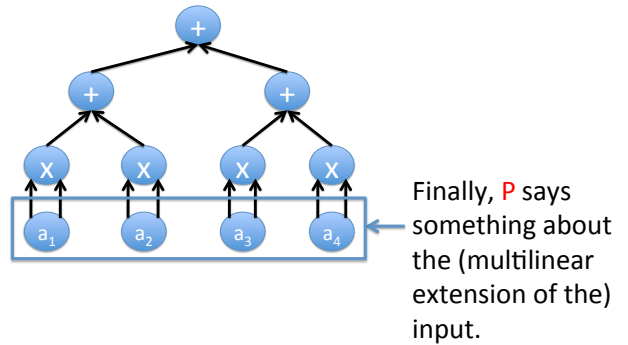


Figure 4.8: In the final iteration, \mathcal{P} makes a claim about the MLE of the input (here, the input of length n with entries in \mathbb{F} is interpreted as a function mapping $\{0, 1\}^{\log_2 n} \rightarrow \mathbb{F}$. Any such function has a unique MLE by Fact 3.5). \mathcal{V} can check this claim without help, since \mathcal{V} sees the input explicitly.

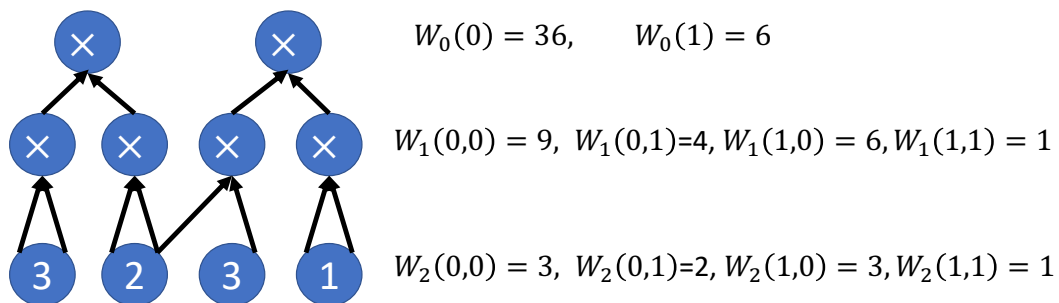


Figure 4.9: Example circuit \mathcal{C} and input x , and resulting functions W_i for each layer i of \mathcal{C} . Note that \mathcal{C} has two output gates.

For an example, consider the circuit depicted in Figure 4.9. Since the circuit contains no addition gates, add_0 and add_1 are the constant 0 function. Meanwhile, mult_0 is the function defined over domain $\{0, 1\} \times \{0, 1\}^2 \times \{0, 1\}^2$ as follows. mult_0 evaluates to 1 on the following two inputs: $(0, (0, 0), (0, 1))$ and $(1, (1, 0), (1, 1))$. On all other inputs, mult_0 evaluates to zero. This is because the first and second in-neighbors of gate 0 at layer 0 are respectively gates $(0, 0)$ and $(0, 1)$ at layer 1, and similarly the first and second in-neighbors of gate 1 at layer 0 are respectively gates $(1, 0)$ and $(1, 1)$ at layer 1.

Similarly, mult_1 is a function on domain $\{0, 1\}^2 \times \{0, 1\}^2 \times \{0, 1\}^2$. It evaluates to 0 on all inputs except for the following four, on which it evaluates to 1:

- $((0, 0), (0, 0), (0, 0))$.
- $((0, 1), (0, 1), (0, 1))$.
- $((1, 0), (0, 1), (1, 0))$.
- $((1, 1), (1, 1), (1, 1))$.

Note that for each layer i , add_i and mult_i depend only on the circuit \mathcal{C} and not on the input x to \mathcal{C} . In contrast, the function \widetilde{W}_i *does* depend on x . This is because \widetilde{W}_i maps each gate label at layer i to the value of the gate when \mathcal{C} is evaluated on input x .

Detailed Description. The GKR protocol consists of d iterations, one for each layer of the circuit. Each iteration i starts with \mathcal{P} claiming a value for $\widetilde{W}_i(r_i)$ for some point in $r_i \in \mathbb{F}^{k_i}$.

At the start of the first iteration, this claim is derived from the claimed outputs of the circuit. Specifically, if there are $S_0 = 2^{k_0}$ outputs of \mathcal{C} , let $D: \{0, 1\}^{k_0} \rightarrow \mathbb{F}$ denote the function that maps the label of an output gate to the claimed value of that output. Then the verifier can pick a random point $r_0 \in \mathbb{F}^{k_0}$ at random, and evaluate $\widetilde{D}(r_0)$ in time $O(S_0)$ using Lemma 3.8. By the Schwartz-Zippel lemma, if $\widetilde{D}(r_0) = \widetilde{W}_0(r_0)$ (i.e., if the multilinear extension of the claimed outputs equals the multilinear extension of the correct outputs when evaluated at a randomly chosen point), then it is safe for the verifier to believe that $\widetilde{D} = \widetilde{W}_0$, and hence that all of the claimed outputs are correct. Unfortunately, the verifier cannot evaluate $\widetilde{W}_0(r_0)$ without help from the prover.

The purpose of iteration i is to reduce the claim about the value of $\widetilde{W}_i(r_i)$ to a claim about $\widetilde{W}_{i+1}(r_{i+1})$ for some $r_{i+1} \in \mathbb{F}^{k_{i+1}}$, in the sense that it is safe for \mathcal{V} to assume that the first claim is true as long as the second claim is true. To accomplish this, the iteration applies the sum-check protocol to a specific polynomial derived from \widetilde{W}_{i+1} , $\widetilde{\text{add}}_i$, and $\widetilde{\text{mult}}_i$. Our description of the protocol actually makes use of a simplification due to Thaler [Tha15].

Applying the Sum-Check Protocol. The GKR protocol exploits an ingenious explicit expression for $\widetilde{W}_i(r_i)$, captured in the following lemma.

Lemma 4.6.

$$\widetilde{W}_i(z) = \sum_{b, c \in \{0, 1\}^{k_{i+1}}} \widetilde{\text{add}}_i(z, b, c) (\widetilde{W}_{i+1}(b) + \widetilde{W}_{i+1}(c)) + \widetilde{\text{mult}}_i(z, b, c) (\widetilde{W}_{i+1}(b) \cdot \widetilde{W}_{i+1}(c)) \quad (4.15)$$

Proof. It is easy to check that the right hand side is a multilinear polynomial in the entries of z , since $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ are multilinear polynomials. (Note that, just as in the matrix multiplication protocol of the Section 4.4, the function being summed over is *quadratic* in the entries of b and c , but this quadratic-ness is “summed away”, leaving a multilinear polynomial only in the variables of z).

Since the multilinear extension of a function with domain $\{0, 1\}^{k_i}$ is unique, it suffices to check that the left hand side and right hand side of the expression in the lemma agree for all $a \in \{0, 1\}^{k_i}$. To this end, fix any $a \in \{0, 1\}^{s_i}$, and suppose that gate a in layer i of \mathcal{C} is an addition gate (the case where gate a is a multiplication gate is similar). Since each gate a at layer i has two unique in-neighbors, namely $\text{in}_1(a)$ and $\text{in}_2(a)$;

$$\text{add}_i(a, b, c) = \begin{cases} 1 & \text{if } (b, c) = (\text{in}_1(a), \text{in}_2(a)) \\ 0 & \text{otherwise} \end{cases}$$

and $\text{mult}_i(a, b, c) = 0$ for all $b, c \in \{0, 1\}^{k_{i+1}}$.

Hence, since $\widetilde{\text{add}}_i$, $\widetilde{\text{mult}}_i$, \widetilde{W}_{i+1} , and \widetilde{W}_i extend add_i and mult_i , W_{i+1} , and W_i respectively,

$$\begin{aligned} & \sum_{b, c \in \{0, 1\}^{k_{i+1}}} \widetilde{\text{add}}_i(a, b, c) (\widetilde{W}_{i+1}(b) + \widetilde{W}_{i+1}(c)) + \widetilde{\text{mult}}_i(a, b, c) (\widetilde{W}_{i+1}(b) \cdot \widetilde{W}_{i+1}(c)) \\ &= \widetilde{W}_{i+1}(\text{in}_1(a)) + \widetilde{W}_{i+1}(\text{in}_2(a)) = W_{i+1}(\text{in}_1(a)) + W_{i+1}(\text{in}_2(a)) = W_i(a) = \widetilde{W}_i(a). \end{aligned}$$

□

Remark 4.3. Lemma 4.6 is actually valid using any extensions of add_i and mult_i that are multilinear in the first k_i variables.

Remark 4.4. Goldwasser, Kalai, and Rothblum [GKR08] use a slightly more complicated expression for $\widetilde{W}_i(a)$ than the one in Lemma 4.6. Their expression allowed them to use even more general extensions of add_i and mult_i (in particular, their extensions do not have to be multilinear in the first k_i variables).

However, the use of the multilinear extensions $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ turns out to be critical to achieving a prover runtime that is nearly *linear* in the circuit size S [CMT12, Tha13], rather than a much larger polynomial in S as achieved by [GKR08] (cf. Section 4.6.5 for details).

Therefore, in order to check the prover's claim about $\widetilde{W}_i(r_i)$, the verifier applies the sum-check protocol to the polynomial

$$f_{r_i}^{(i)}(b, c) = \widetilde{\text{add}}_i(r_i, b, c) (\widetilde{W}_{i+1}(b) + \widetilde{W}_{i+1}(c)) + \widetilde{\text{mult}}_i(r_i, b, c) (\widetilde{W}_{i+1}(b) \cdot \widetilde{W}_{i+1}(c)). \quad (4.16)$$

Note that *the verifier does not know the polynomial \widetilde{W}_{i+1}* (as this polynomial defined in terms of gate values at layer $i + 1$ of the circuit, and unless $i + 1$ is the input layer, the verifier does not have direct access to the values of these gates), and hence the verifier does not actually know the polynomial $f_{r_i}^{(i)}$ that it is applying the sum-check protocol to. Nonetheless, it is possible for the verifier to apply the sum-check protocol to $f_{r_i}^{(i)}$ because, until the final round, the sum-check protocol does not require the verifier to know anything about the polynomial other than its degree in each variable (see Remark 4.2). However, there remains the issue that \mathcal{V} can only execute the final check in the sum-check protocol if she can evaluate the polynomial $f_{r_i}^{(i)}$ at a random point. This is handled as follows.

Let us denote the random point at which \mathcal{V} must evaluate $f_{r_i}^{(i)}$ by (b^*, c^*) , where $b^* \in \mathbb{F}^{k_{i+1}}$ is the first k_{i+1} entries and $c^* \in \mathbb{F}^{k_{i+1}}$ the last k_{i+1} entries. Note that b^* , and c^* may have non-Boolean entries. Evaluating $f_{r_i}^{(i)}(b^*, c^*)$ requires evaluating $\widetilde{\text{add}}_i(r_i, b^*, c^*)$, $\widetilde{\text{mult}}_i(r_i, b^*, c^*)$, $\widetilde{W}_{i+1}(b^*)$, and $\widetilde{W}_{i+1}(c^*)$.

For many circuits, particularly those whose wiring pattern display repeated structure, \mathcal{V} can evaluate $\widetilde{\text{add}}_i(r_i, b^*, c^*)$ and $\widetilde{\text{mult}}_i(r_i, b^*, c^*)$ on her own in $O(k_i + k_{i+1})$ time as well. For now, assume that \mathcal{V} can indeed perform this evaluation in $\text{poly}(k_i, k_{i+1})$ time, but this issue will be discussed further in Section 4.6.6.

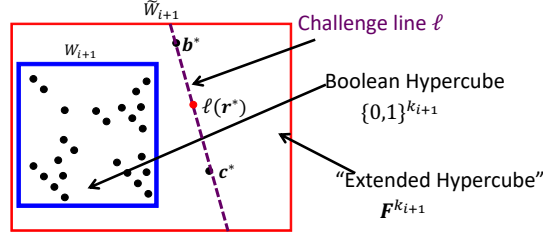


Figure 4.10: Schematic of how to reduce verifying claims about the values of $\tilde{W}_{i+1}(b^*)$ and $\tilde{W}_{i+1}(c^*)$ to a single claim about the value of $\tilde{W}_{i+1}(r_{i+1})$. Here, if ℓ is the unique line passing through b^* and c^* , then r_{i+1} is a random point on ℓ .

\mathcal{V} cannot however evaluate $\tilde{W}_{i+1}(b^*)$, and $\tilde{W}_{i+1}(c^*)$ on her own without evaluating the circuit. Instead, \mathcal{V} asks \mathcal{P} to simply provide these two values, say, z_1 and z_2 , and uses iteration $i + 1$ to *verify* that these values are as claimed. However, one complication remains: the precondition for iteration $i + 1$ is that \mathcal{P} claims a value for $\tilde{W}_{i+1}(r_{i+1})$ for a single point $r_{i+1} \in \mathbb{F}^{k_{i+1}}$. So \mathcal{V} needs to reduce verifying both $\tilde{W}_{i+1}(b^*) = z_1$ and $\tilde{W}_{i+1}(c^*) = z_2$ to verifying $\tilde{W}_{i+1}(r_{i+1})$ at a single point $r_{i+1} \in \mathbb{F}^{k_{i+1}}$, in the sense that it is safe for \mathcal{V} to accept the claimed values of $\tilde{W}_{i+1}(b^*)$ and $\tilde{W}_{i+1}(c^*)$ as long as the value of $\tilde{W}_{i+1}(r_{i+1})$ is as claimed. This is done as follows.

Reducing to Verification of a Single Point. Let $\ell : \mathbb{F} \rightarrow \mathbb{F}^{k_{i+1}}$ be some canonical line passing through b^* and c^* . For example, we can let $\ell : \mathbb{F} \rightarrow \mathbb{F}^{k_{i+1}}$ be the unique line such that $\ell(0) = b^*$ and $\ell(1) = c^*$. \mathcal{P} sends a univariate polynomial q of degree at most k_{i+1} that is claimed to be $\tilde{W}_{i+1} \circ \ell$, the restriction of \tilde{W}_{i+1} to the line ℓ . \mathcal{V} checks that $q(0) = z_1$ and $q(1) = z_2$ (rejecting if this is not the case), picks a random point $r^* \in \mathbb{F}$, and asks \mathcal{P} to prove that $\tilde{W}_{i+1}(\ell(r^*)) = q(r^*)$. By the Schwartz-Zippel Lemma (even its simple special case for univariate polynomials), as long as \mathcal{V} is convinced that $\tilde{W}_{i+1}(\ell(r^*)) = q(r^*)$, it is safe for \mathcal{V} to believe that q does in fact equal $\tilde{W}_{i+1} \circ \ell$, and hence that $\tilde{W}_{i+1}(b^*) = z_1$ and $\tilde{W}_{i+1}(c^*) = z_2$ as claimed by \mathcal{P} . This completes iteration i ; \mathcal{P} and \mathcal{V} then move on to the iteration for layer $i + 1$ of the circuit, whose purpose is to verify that $\tilde{W}_{i+1}(r_{i+1})$ has the claimed value, where $r_{i+1} := \ell(r^*)$.

A picture and an example. This technique is depicted pictorially in Figure 4.10. For a concrete example of how this technique works, suppose that $k_{i+1} = 2$, $b^* = (2, 4)$, $c^* = (3, 2)$, and $\tilde{W}_{i+1}(x_1, x_2) = 3x_1x_2 + 2x_2$. Then the unique line $\ell(t)$ with $\ell(0) = b^*$ and $\ell(1) = c^*$ is $t \mapsto (t + 2, 4 - 2t)$. The restriction of \tilde{W}_{i+1} to ℓ is $3(t + 2)(4 - 2t) + 2(4 - 2t) = -6t^2 - 4t + 32$. If \mathcal{P} sends a degree-2 univariate polynomial q claimed to equal $\tilde{W}_{i+1} \circ \ell$, the verifier will interpret $q(0)$ and $q(1)$ as claims about the values $\tilde{W}_{i+1}(b^*)$ and $\tilde{W}_{i+1}(c^*)$ respectively. The verifier will then pick a random $r^* \in \mathbb{F}$. Observe that $\ell(r^*) = (r^* + 2, 4 - 2r^*)$ is a random point on the line ℓ . Iteration $i + 1$ of the GKR protocol is devoted to confirming that $\tilde{W}_{i+1}(r^* + 2, 4 - 2r^*) = q(r^*)$.

The Final Iteration. Finally, at the final iteration d , \mathcal{V} must evaluate $\tilde{W}_d(r_d)$ on her own. But the vector of gate values at layer d of \mathcal{C} is simply the input x to \mathcal{C} . By Lemma 3.8, \mathcal{V} can compute $\tilde{W}_d(r_d)$ on her own in $O(n)$ time.

A self-contained description of the GKR protocol is provided in Figure 4.11.

4.6.5 Discussion of Costs

\mathcal{V} 's runtime. Observe that the polynomial $f_{r_i}^{(i)}$ defined in Equation (4.16) is a $(2k_{i+1})$ -variate polynomial of degree at most 2 in each variable, and so the invocation of the sum-check protocol at iteration i requires $2k_{i+1}$

Description of the GKR protocol, when applied to a layered arithmetic circuit \mathcal{C} of depth d and fan-in two on input $x \in \mathbb{F}^n$. Throughout, k_i denotes $\log_2(S_i)$ where S_i is the number of gates at layer i of \mathcal{C} .

- At the start of the protocol, \mathcal{P} sends a function $D: \{0, 1\}^{k_0} \rightarrow \mathbb{F}$ claimed to equal W_0 (the function mapping output gate labels to output values).
- \mathcal{V} picks a random $r_0 \in \mathbb{F}^{k_0}$ and lets $m_0 \leftarrow \tilde{D}(r_0)$. The remainder of the protocol is devoted to confirming that $m_0 = \tilde{W}_0(r_0)$.
- **For** $i = 0, 1, \dots, d - 1$:

- Define the $(2k_{i+1})$ -variate polynomial

$$f_{r_i}^{(i)}(b, c) := \widetilde{\text{add}}_i(r_i, b, c) (\tilde{W}_{i+1}(b) + \tilde{W}_{i+1}(c)) + \widetilde{\text{mult}}_i(r_i, b, c) (\tilde{W}_{i+1}(b) \cdot \tilde{W}_{i+1}(c)).$$

- \mathcal{P} claims that $\sum_{b, c \in \{0, 1\}^{k_{i+1}}} f_{r_i}^{(i)}(b, c) = m_i$.
- So that \mathcal{V} may check this claim, \mathcal{P} and \mathcal{V} apply the sum-check protocol to $f_{r_i}^{(i)}$, up until \mathcal{V} 's final check in that protocol, when \mathcal{V} must evaluate $f_{r_i}^{(i)}$ at a randomly chosen point $(b^*, c^*) \in \mathbb{F}^{k_{i+1}} \times \mathbb{F}^{k_{i+1}}$. See Remark (a) at the end of this codebox.
- Let ℓ be the unique line satisfying $\ell(0) = b^*$ and $\ell(1) = c^*$. \mathcal{P} sends a univariate polynomial q of degree at most k_{i+1} to \mathcal{V} , claimed to equal \tilde{W}_{i+1} restricted to ℓ .
- \mathcal{V} now performs the final check in the sum-check protocol, using $q(0)$ and $q(1)$ in place of $\tilde{W}_{i+1}(b^*)$ and $\tilde{W}_{i+1}(c^*)$. See Remark (b) at the end of this codebox.
- \mathcal{V} chooses $r^* \in \mathbb{F}$ at random and sets $r_{i+1} = \ell(r^*)$ and $m_{i+1} \leftarrow q(r_{i+1})$.

- \mathcal{V} checks directly that $m_d = \tilde{W}_d(r_d)$ using Lemma 3.8.

Note that \tilde{W}_d is simply \tilde{x} , the multilinear extension of the input x when x is interpreted as the evaluation table of a function mapping $\{0, 1\}^{\log n} \rightarrow \mathbb{F}$.

Remark a. Note that \mathcal{V} does not actually know the polynomial $f_{r_i}^{(i)}$, because \mathcal{V} does not know the polynomial \tilde{W}_{i+1} that appears in the definition of $f_{r_i}^{(i)}$. However, the sum-check protocol does not require \mathcal{V} to know anything about the polynomial to which it is being applied, until the very final check in the protocol (see Remark 4.2).

Remark b. We assume here that for each layer i of \mathcal{C} , \mathcal{V} can evaluate the multilinear extensions $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ at the point (r_i, b^*, c^*) in polylogarithmic time. Hence, given $\tilde{W}_{i+1}(b^*)$ and $\tilde{W}_{i+1}(c^*)$, \mathcal{V} can quickly evaluate $f_{r_i}^{(i)}(b^*, c^*)$ and thereby perform its final check in the sum-check protocol applied to $f_{r_i}^{(i)}$.

Figure 4.11: Self-contained description of the GKR protocol for arithmetic circuit evaluation.

rounds, with three field elements transmitted per round. Thus, the total communication cost is $O(S_0 + d \log S)$ field elements where S_0 is the number of outputs of the circuit. The time cost to \mathcal{V} is $O(n + d \log S + t + S_0)$, where t is the amount of time required for \mathcal{V} to evaluate $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ at a random input, for each layer i of \mathcal{C} . Here the n term is due to the time required to evaluate $\widetilde{W}_d(r_d)$, the S_0 term is the time required to read the vector of claimed outputs and evaluate the corresponding multilinear extension, the $d \log S$ term is the time required for \mathcal{V} to send messages to \mathcal{P} and process and check the messages from \mathcal{P} . For now, let us assume that t is a low-order cost and that $S_0 = 1$, so that \mathcal{V} runs in total time $O(n + d \log S)$; we discuss this issue further in Section 4.6.6.

\mathcal{P} 's runtime. Analogously to the MATMULT protocol of Section 4.4, we give two increasingly sophisticated implementations of the prover when the sum-check protocol is applied to the polynomial $f_{r_i}^{(i)}$.

Method 1: $f_{r_i}^{(i)}$ is a v -variate polynomial for $v = 2k_{i+1}$. As in the analysis of Method 1 for implementing the prover in the matrix multiplication protocol from Section 4.4, \mathcal{P} can compute the prescribed method in round j by evaluating $f_{r_i}^{(i)}$ at $3 \cdot 2^{v-j}$ points. It is not hard to see that \mathcal{P} can evaluate $f_{r_i}^{(i)}$ at any point in $O(S_i + S_{i+1})$ time using techniques similar to Lemma 3.8. This yields a runtime for \mathcal{P} of $O(2^v \cdot (S_i + S_{i+1}))$. Over all d layers of the circuit, \mathcal{P} 's runtime is bounded by $O(S^3)$.

Method 2: Cormode et al. [CMT12] improved on the $O(S^3)$ runtime of Method 1 by observing, just as in the matrix multiplication protocol from Section 4.4, that the $3 \cdot 2^{v-j}$ points at which \mathcal{P} must evaluate $f_{r_i}^{(i)}$ in round j of the sum-check protocol are highly structured, in the sense that their trailing entries are Boolean. That is, it suffices for \mathcal{P} to evaluate $f_{r_i}^{(i)}(z)$ for all points z of the form: $z = (r_1, \dots, r_{j-1}, \{0, 1, 2\}, b_{j+1}, \dots, b_v)$, where $v = 2k_{i+1}$ and each $b_k \in \{0, 1\}$.

For each such point z , the bottleneck in evaluating $f_{r_i}^{(i)}(z)$ is in evaluating $\widetilde{\text{add}}_i(z)$ and $\widetilde{\text{mult}}_i(z)$. A direct application of Lemma 3.8 implies that each such evaluation can be performed in $2^v = O(S_{i+1}^2)$ time. However, we can do much better by observing that the functions add_i and mult_i are *sparse*, in the sense that $\text{add}_i(a, b, c) = \text{mult}_i(a, b, c) = 0$ for all Boolean vectors $(a, b, c) \in \mathbb{F}^v$ except for the S_i vectors of the form $(a, \text{in}_{1,i}(a), \text{in}_{2,i}(a)) : a \in \{0, 1\}^{k_i}$.

Thus, by Lagrange Interpolation (Lemma 3.6), we can write $\widetilde{\text{add}}_i(z) = \sum_{a \in \{0, 1\}^{k_i}} \chi_{(a, \text{in}_{1,i}(a), \text{in}_{2,i}(a))}(z)$, where the sum is only over addition gates a at layer i of \mathcal{C} , and similarly for $\widetilde{\text{mult}}_i(z)$ (recall that the multilinear Lagrange basis polynomial $\chi_{(a, \text{in}_{1,i}(a), \text{in}_{2,i}(a))}$ was defined in Equation (3.2) of Lemma 3.6). Just as in the analysis of Method 2 for implementing the prover in the matrix multiplication protocol of Section 4.4, for any input z of the form $z = (r_1, \dots, r_{j-1}, \{0, 1, 2\}, b_{j+1}, \dots, b_v)$, it holds that $\chi_{(a, \text{in}_{1,i}(a), \text{in}_{2,i}(a))}(z) = 0$ unless the last $v - j$ entries of z and $(a, \text{in}_{1,i}(a), \text{in}_{2,i}(a))$ are equal (here, we are exploiting the fact that the trailing entries of z are Boolean). Hence, \mathcal{P} can evaluate $\widetilde{\text{add}}_i(z)$ at all the necessary points z in each round of the sum-check protocol with a single pass over the gates at layer i of \mathcal{C} : for each gate a in layer i , \mathcal{P} only needs to update $\widetilde{\text{add}}_i(z) \leftarrow \widetilde{\text{add}}_i(z) + \chi_{(a, \text{in}_{1,i}(a), \text{in}_{2,i}(a))}(z)$ for the three values of z whose trailing $v - j$ entries equal the trailing entries of $(a, \text{in}_{1,i}(a), \text{in}_{2,i}(a))$.

Round Complexity and Communication Cost. By direct inspection of the protocol description, there are $O(d \log S)$ rounds in the GKR protocol, and the total communication cost is $O(d \log S)$ field elements.

Soundness error. The soundness error of the GKR protocol is $O(d \log(S)/|\mathbb{F}|)$. The idea of the soundness analysis is that, if the prover begins the protocol with a false claim as to the output value(s) $\mathcal{C}(x)$, then for the verifier to be convinced to accept, there must be at least one round j of the interactive proof in which the following occurs. The prover sends a univariate polynomial g_j that differs from the prescribed polynomial s_j

that the honest prover would have sent in that round, yet $g_j(r_j) = s_j(r_j)$, where r_j is a random field element chosen by the verifier in round j . For rounds j of the GKR protocol corresponding to a round within an invocation of the sum-check protocol, g_j and s_j are polynomials of degree $O(1)$, and hence if $g_j \neq s_j$ then the probability (over the random choice of r_j) that $g_j(r_j) = s_j(r_j)$ is at most $O(1/|\mathbb{F}|)$.

In rounds j of the GKR protocol corresponding to the “reducing to verification of a single point” technique (depicted in Figure 4.10), g_j and s_j have degree at most $O(\log S)$, and hence if $g_j \neq s_j$, the probability that $g_j(r_j) = s_j(r_j)$ is at most $O(\log(S)/|\mathbb{F}|)$. Note that there are at most d such rounds over the course of the entire protocol, since this technique is applied at most once per layer of \mathcal{C} .

By applying a union bound over all rounds in the protocol, we conclude that the probability there is *any* round j such that $g_j \neq s_j$ yet $g_j(r_j) = s_j(r_j)$ is at most $O(d \log(S)/|\mathbb{F}|)$.

4.6.6 Evaluating $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ Efficiently

The issue of the verifier efficiently evaluating $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ at a random point $\omega \in \mathbb{F}^{k_i+2k_{i+1}}$ is a tricky one. While there does not seem to be a clean characterization of precisely which circuits have $\widetilde{\text{add}}_i$'s and $\widetilde{\text{mult}}_i$'s that can be evaluated in $O(\log S)$ time, most circuits that exhibit any kind of repeated structure satisfies this property. In particular, the papers [CMT12, Tha13] show that the evaluation can be computed in $O(k_i + k_{i+1}) = O(\log S)$ time for a variety of common wiring patterns and specific circuits. This includes the canonical circuit for simulating a space-bounded machine whose construction was sketched in Section 4.6.2. It also includes where the wiring patterns involve basic arithmetic on gate indices, and specific circuits computing functions such as MATMULT, pattern matching, Fast Fourier Transforms, and various problems of interest in the streaming literature, like frequency moments and distinct elements (see Exercise 4.4). In a similar vein, Holmgren and Rothblum [HR18, Section 5.1] show that as long as $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ are computable within a computational model called read-once branching programs, then $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ can be evaluated at any desired point in logarithmic time, and observe that this condition indeed captures common wiring patterns. Moreover, we will see in Section 4.6.7 that $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ can be evaluated efficiently for any circuit that operates in a *data parallel* manner.

In addition, various suggestions have been put forth for what to do when $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ cannot be evaluated in time $O(\log S)$. For example, as observed by Cormode et al. [CMT12], these computations can always be done by \mathcal{V} in $O(\log S)$ space as long as the circuit is log-space uniform. This is sufficient in streaming applications where the space usage of the verifier is paramount [CMT12]. Moreover, these computations can be done offline before the input is even observed, because they only depend on the wiring of the circuit, and not on the input [GKR08, CMT12].

An additional proposal appeared in [GKR08], where Goldwasser, Kalai, and Rothblum considered the option of outsourcing the computation of $\widetilde{\text{add}}_i(r_i, b^*, c^*)$ and $\widetilde{\text{mult}}_i(r_i, b^*, c^*)$ themselves. In fact, this option plays a central role in obtaining their result for general log-space uniform circuits. Specifically, GKR's results for general log-space uniform circuits are obtained via a two-stage protocol. First, they give a protocol for any problem computable in (non-deterministic) logarithmic space by applying their protocol to the canonical circuit for simulating a space-bounded Turing machine. This circuit has a highly regular wiring pattern for which $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ can be evaluated in $O(\log S)$ time.⁴⁵ For a general log-space uniform circuit \mathcal{C} , it is not known how to identify low-degree extensions of $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ that can be evaluated at ω in polylogarithmic time. Rather, Goldwasser et al. outsource computation of $\widetilde{\text{add}}_i(r_i, b^*, c^*)$ and $\widetilde{\text{mult}}_i(r_i, b^*, c^*)$

⁴⁵In [GKR08], Goldwasser et al. actually use higher degree extensions of $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ obtained by arithmetizing a Boolean formula of size $\text{polylog}(S)$ computing these functions. The use of these extensions results in a prover whose runtime is a large polynomial in S (i.e., $O(S^4)$). Cormode et al. [CMT12] observe that in fact the multilinear extensions of $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ can be used for this circuit, and that with these extensions the prover's runtime can be brought down to $O(S \log S)$.

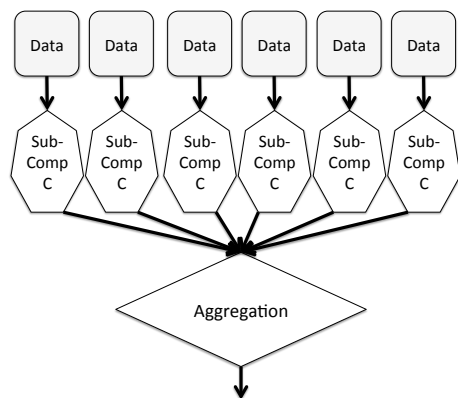


Figure 4.12: Schematic of a data parallel computation.

themselves. Since \mathcal{C} is log-space uniform, $\widetilde{\text{add}}_i(r_i, b^*, c^*)$ and $\widetilde{\text{mult}}_i(r_i, b^*, c^*)$ can be computed in logarithmic space, and the protocol for logspace computations applies directly.

A closely related proposal to deal with the circuits for which $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ cannot be evaluated in time sublinear in the circuit size S leverages cryptography. Specifically, later in this manuscript we will introduce a cryptographic primitive called *polynomial commitment scheme* and explain how this primitive can be used to achieve the following. A trusted party (e.g., the verifier itself) can spend $\widetilde{O}(S)$ time in pre-processing and produce a short *cryptographic commitment* to the polynomials $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ for all layers i of \mathcal{C} . After this pre-processing stage, the verifier \mathcal{V} can apply the IP of this section to evaluate \mathcal{C} on many different inputs, and \mathcal{V} can use the cryptographic commitment to force the prover to accurately evaluate $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ on its behalf. Due to its use of cryptography, this proposal results in an argument system as opposed to an interactive proof. Argument systems that handle pre-processing in this manner are sometimes called *holographic* in the research literature. See Section 13.3 for details.

4.6.7 Leveraging Data Parallelism for Further Speedups

Data parallel computation refers to any setting in which the same sub-computation is applied independently to many pieces of data, before possibly aggregating the results. The protocol of this section makes no assumptions on the sub-computation that is being applied (in particular, it handles sub-computations computed by circuits with highly irregular wiring patterns), but does assume that the sub-computation is applied independently to many pieces of data. Figure 4.12 gives a schematic of a data parallel computation.

Data parallel computation is pervasive in real-world computing. For example, consider any *counting query* on a database. In a counting query, one applies some function independently to each row of the database and sums the results. For example, one may ask “How many people in the database satisfy Property P ?” The protocol below allows one to verifiably outsource such a counting query with overhead that depends minimally on the size of the database, but that necessarily depends on the complexity of the property P . In Section 5.5, we will see that data parallel computations are in some sense “universal”, in that efficient transformations from high-level computer programs to circuits often yield data parallel circuits.

The Protocol and its Costs. Let C be a circuit of size S with an arbitrary wiring pattern, and let C' be a “super-circuit” that applies C independently to $B = 2^b$ different inputs before aggregating the results in some fashion. For example, in the case of a counting query, the aggregation phase simply sums the results of the

data parallel phase. Assume that the aggregation step is sufficiently simple that the aggregation itself can be verified using the techniques of Section 4.6.5.

If one naively applies the GKR protocol to the super-circuit C' , \mathcal{V} might have to perform an expensive pre-processing phase to evaluate the wiring predicates add_i and mult_i of C' at the necessary locations—this would require time $\Omega(B \cdot S)$. Moreover, when applying the basic GKR protocol to C' using the techniques of [CMT12], \mathcal{P} would require time $\Theta(B \cdot S \cdot \log(B \cdot S))$. A different approach was taken by Vu et al. [VSBW13], who applied the GKR protocol B independent times, once for each copy of C . This causes both the communication cost and \mathcal{V} 's online check time to grow linearly with B , the number of sub-computations, which is undesirable.

In contrast, the protocol of this section (due to [WJB⁺17], building on [Tha13]) achieves the best of both worlds, in that the overheads for the prover and verifier have no dependence on the number of inputs B to which C is applied. More specifically, the preprocessing time of the verifier is at most $O(S)$, independent of B . The prover runs in time $O(BS + S \log S)$. Observe that as long as $B > \log S$ (i.e., there is a sufficient amount of data parallelism in the computation), $O(BS + S \log S) = O(B \cdot S)$, and hence the prover is only a constant factor slower than the time required to evaluate the circuit gate-by-gate with no guarantee of correctness.

The idea of the protocol is that although each sub-computation C can have a complicated wiring pattern, the circuit is maximally regular between sub-computations, as the sub-computations do not interact at all. It is possible to leverage this regularity to minimize the pre-processing time of the verifier, and to significantly speed up the prover.

4.6.7.1 Protocol Details

Let C be an arithmetic circuit over \mathbb{F} of depth d and size S with an arbitrary wiring pattern, and let C' be the circuit of depth d and size $B \cdot S$ obtained by laying B copies of C side-by-side, where $B = 2^b$ is a power of 2. We will use the same notation as in Section 4.6.4, using apostrophes to denote quantities referring to C' . For example, layer i of C has size $S_i = 2^{k_i}$ and gate values specified by the function W_i , while layer i of C' has size $S'_i = 2^{k'_i} = 2^{b+k_i}$ and gate values specified by W'_i .

Consider layer i of C' . Let $a = (a_1, a_2) \in \{0, 1\}^{k_i} \times \{0, 1\}^b$ be the label of a gate at layer i of C' , where a_2 specifies which “copy” of C the gate is in, while a_1 designates the label of the gate within the copy. Similarly, let $b = (b_1, b_2) \in \{0, 1\}^{k_{i+1}} \times \{0, 1\}^b$ and $c = (c_1, c_2) \in \{0, 1\}^{k_{i+1}} \times \{0, 1\}^b$ be the labels of two gates at layer $i+1$. The key to achieving the speedups for data parallel circuits relative to the interactive proof described in Section 4.6.4 is to tweak the expression in Lemma 4.6 for \tilde{W}_i . Specifically, Lemma 4.6 represents $\tilde{W}'_i(z)$ as a sum over $(S'_{i+1})^2$ terms. In this section, we leverage the data parallel structure of C' to represent $\tilde{W}'_i(z)$ as a sum over $S'_{i+1} \cdot S_{i+1}$ terms, which is smaller than $(S'_{i+1})^2$ by a factor of B .

Lemma 4.7. *Let h denote the polynomial $\mathbb{F}^{k_i \times b} \rightarrow \mathbb{F}$ defined via*

$$h(a_1, a_2) := \sum_{b_1, c_1 \in \{0, 1\}^{k_{i+1}}} g(a_1, a_2, b_1, c_1),$$

where

$$g(a_1, a_2, b_1, c_1) := \widetilde{\text{add}}_i(a_1, b_1, c_1) \left(\tilde{W}'_{i+1}(b_1, a_2) + \tilde{W}'_{i+1}(c_1, a_2) \right) + \widetilde{\text{mult}}_i(a_1, b_1, c_1) \cdot \tilde{W}'_{i+1}(b_1, a_2) \cdot \tilde{W}'_{i+1}(c_1, a_2).$$

Then h extends W'_i .

Proof Sketch. Essentially, Lemma 4.7 says that an addition (respectively, multiplication) gate $a = (a_1, a_2) \in \{0, 1\}^{k_i+b}$ of \mathcal{C}' is connected to gates $b = (b_1, b_2) \in \{0, 1\}^{k_{i+1}+b}$ and $c = (c_1, c_2) \in \{0, 1\}^{k_{i+1}+b}$ of \mathcal{C}' if and only if a , b , and c are all in the same copy of \mathcal{C} , and a is connected to b and c within the copy. \square

The following lemma requires some additional notation. Let $\beta_{k'_i}(a, b) : \{0, 1\}^{k'_i} \times \{0, 1\}^{k'_i} \rightarrow \{0, 1\}$ be the function that evaluates to 1 if $a = b$, and evaluates to 0 otherwise, and define the formal polynomial

$$\tilde{\beta}_{k'_i}(a, b) = \prod_{j=1}^{k'_i} ((1 - a_j)(1 - b_j) + a_j b_j). \quad (4.17)$$

It is straightforward to check that $\tilde{\beta}_{k'_i}$ is the multilinear extension $\beta_{k'_i}$. Indeed, $\tilde{\beta}_{k'_i}$ is a multilinear polynomial. And for $a, b \in \{0, 1\}^{k'_i}$, it is easy to check that $\tilde{\beta}_{k'_i}(a, b) = 1$ if and only if a and b are equal coordinate-wise.

Lemma 4.8. (Restatement of [Rot09, Lemma 3.2.1].) For any polynomial $h : \mathbb{F}^{k'_i} \rightarrow \mathbb{F}$ extending W'_i , the following polynomial identity holds:

$$\tilde{W}'_i(z) = \sum_{a \in \{0, 1\}^{k'_i}} \tilde{\beta}_{k'_i}(z, a) h(a). \quad (4.18)$$

Proof. It is easy to check that the right hand side of Equation (4.18) is a multilinear polynomial in z , and that it agrees with W'_i on all Boolean inputs. Thus, the right hand side of Equation (4.18), viewed as a polynomial in z , must be the (unique) multilinear extension \tilde{W}'_i of W'_i . \square

Combining Lemmas 4.7 and 4.8 implies that for any $z \in \mathbb{F}^{k'_i}$,

$$\tilde{W}'_i(z) = \sum_{(a_1, a_2, b_1, c_1) \in \{0, 1\}^{k_i+b+2k_{i+1}}} g_z^{(i)}(a_1, a_2, b_1, c_1), \quad (4.19)$$

where

$$g_z^{(i)}(a_1, a_2, b_1, c_1) := \tilde{\beta}_{k'_i}(z, (a_1, a_2)) \cdot \left[\widetilde{\text{add}}_i(a_1, b_1, c_1) \left(\tilde{W}'_{i+1}(b_1, a_2) + \tilde{W}'_{i+1}(c_1, a_2) \right) + \widetilde{\text{mult}}_i(a_1, b_1, c_1) \cdot \tilde{W}'_{i+1}(b_1, a_2) \cdot \tilde{W}'_{i+1}(c_1, a_2) \right].$$

Thus, to reduce a claim about $\tilde{W}'_i(r_i)$ to a claim about $\tilde{W}'_{i+1}(r_{i+1})$ for some point $r_{i+1} \in \mathbb{F}^{k'_{i+1}}$, it suffices to apply the sum-check protocol to the polynomial $g_{r_i}^{(i)}$, and then use the ‘‘Reducing to Verification of a Single Point’’ technique from Section 4.6.4. That is, the protocol is the same as in Section 4.6.4, except that, at layer i , rather than applying the sum-check protocol to the polynomial $f_{r_i}^{(i)}$ defined in Equation (4.16) to compute $\tilde{W}'_i(r_i)$, the protocol instead applies the sum-check protocol to the polynomial $g_{r_i}^{(i)}$ (Equation (4.19)).

Costs for \mathcal{V} . To bound \mathcal{V} ’s runtime, observe that $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ can be evaluated at a random point in $\mathbb{F}^{k_i+2k_{i+1}}$ in pre-processing in time $O(S_i)$ by enumerating the in-neighbors of each of the S_i gates at layer i in order to apply Lemma 3.8. Adding up the pre-processing time across all iterations i of our protocol, \mathcal{V} ’s pre-processing time is $O(\sum_i S_i) = O(S)$ as claimed. Notice this pre-processing time is independent of B , the number of copies of the subcircuit.

Outside of pre-processing, the costs to the verifier are similar to Section 4.6.5, with the main difference being that now the verifier needs to also evaluate $\tilde{\beta}_{k'_i}$ at a random point at each layer i . But the verifier can

evaluate $\tilde{\beta}_{k_i}$ at any input with $O(\log S_i)$ additions and multiplications over \mathbb{F} , using Equation (4.17). This does not affect the verifier’s asymptotic runtime.

Costs for \mathcal{P} . The insights that go into implementing the honest prover in time $O(B \cdot S + S \log S)$ build on ideas related the Method 3 for implementing the prover in the Matrix Multiplication protocol of Section 4.4, and heavily exploit the fact that Equation (4.19) represents $\tilde{W}'_i(z)$ as a sum over just $S'_{i+1} \cdot S_{i+1}$ terms, rather than the $(S'_{i+1})^2$ terms in the sum that would be obtained by applying Equation (4.15) to C' .

Remark 4.5. Recent work [XZZ⁺19] has shown how to use Lemma 4.5 to implement the prover in the IP of Section 4.6.4 in time $O(S)$ for *arbitrary* arithmetic circuits of size S (not just circuits with a sufficient amount of data parallelism as in Section 4.6.7).⁴⁶ For brevity, we do not elaborate here upon how to achieve this result. The same result in fact follows (with some adaptation) from Section 7.5 in Chapter 7, where we explain how to achieve an $O(S)$ -time prover in a (two-prover) interactive proof for a *generalization* of arithmetic circuits, called *rank-one constraint systems* (R1CS).

4.6.8 Tension Between Efficiency and Generality

The GKR protocol and its variants covered in this chapter is an example of a *general-purpose* technique for designing VC protocols. Specifically, the GKR protocol can be used to verifiably outsource the evaluation of an arbitrary arithmetic circuit (and as we will see in the next chapter, arbitrary computer programs can be turned into arithmetic circuits). Such general-purpose techniques are the primary focus of this survey.

However, there is often a tension between the generality and efficiency of VC protocols. That is, the general-purpose techniques should sometimes be viewed as heavy hammers that are capable of pounding arbitrary nails, but are not necessarily the most efficient way of hammering any particular nail.

This point was already raised in Section 4.4.1 in the context of matrix multiplication (see the paragraph “Preview: Other Protocols for Matrix Multiplication”). That section described an interactive proof for matrix multiplication that is far more concretely efficient, especially in terms of prover time and communication cost, than applying the GKR protocol to any known arithmetic circuit computing matrix multiplication. As another example, the circuit depicted in Figures 4.5-4.8 computes the sum of the squared entries of the input in \mathbb{F}^n . This is an important function in the literature on streaming algorithms, called the second frequency moment. Applying the GKR protocol to this circuit (which has logarithmic depth and size $O(n)$) would result in communication cost of $\Theta(\log^2 n)$. But the function can be computed much more directly, and with total communication $O(\log n)$, by a single application of the sum-check protocol. Specifically, if we interpret the input as specifying a function $f: \mathbb{F}^{\log n} \rightarrow \mathbb{F}$ in the natural way, then we can simply apply the sum-check protocol to the polynomial $(\tilde{f})^2$, the square of the multilinear extension of f . This requires the verifier to evaluate $(\tilde{f})^2$ at a single point r . The verifier can compute $(\tilde{f})^2(r)$ by evaluating $\tilde{f}(r)$ in linear or quasilinear time using Lemma 3.7 or Lemma 3.8, and then squaring the result.

To summarize, while this survey is primarily focused on general-purpose VC protocols, these do not represent the most efficient solutions in all situations. Those interested in specific functionalities may be well-advised to consider whether less general but more efficient protocols apply to the functionality of interest. Even when using a general-purpose VC protocol, there are typically many optimizations a protocol designer can identify (e.g., expanding the gate set within the GKR protocol from addition and multiplication gates to other types of low-degree operations tailored to the functionality of interest, see for example [CMT12, Section 3.2], [XZZ⁺19, Section 5], and [BB20]).

⁴⁶To clarify, this does not address the issue discussed in Section 4.6.6 that for arbitrary arithmetic circuits, the verifier may need time linear in the circuit size S to evaluate add_i and mult_i as required by the protocol.

Communication	Rounds	\mathcal{V} time	\mathcal{P} time
$O(d \cdot \log(B \cdot S))$ field elements	$O(d \cdot (\log(B \cdot S)))$	online time: $O(B \cdot n + d \cdot (\log(B \cdot S)))$ pre-processing time: $O(S)$	$O(B \cdot S + S \cdot \log(S))$

Table 4.5: Costs of the IP of Section 4.6.7 when applied to any log-space uniform arithmetic circuit \mathcal{C} of size S and depth d over n variables, that is applied B times in a data parallel manner (cf. Figure 4.12).

4.7 Publicly Verifiable, Non-interactive Argument via Fiat-Shamir

4.7.1 The Random Oracle Model

The random oracle model (ROM) [FS86, BR93] is an idealized setting meant to capture the fact that cryptographers have developed hash functions (e.g., SHA-3 or BLAKE3) that efficient algorithms seem totally unable to distinguish from random functions. By a random function R mapping some domain \mathcal{D} to the κ -bit range $\{0, 1\}^\kappa$, we mean the following: on any input $x \in \mathcal{D}$, R chooses its output $R(x)$ uniformly at random from $\{0, 1\}^\kappa$.

Accordingly, the ROM simply assumes that the prover and verifier have query access to a random function R . This means that there is an oracle (called a random oracle) such that the prover and verifier can submit any query x to the oracle, and the oracle will return $R(x)$. That is, for each query $x \in \mathcal{D}$ posed to the oracle, the oracle makes an independent random choice to determine $R(x)$ and responds with that value. It keeps a record of its responses to make sure that it repeats the same response if x is queried again.

The random oracle assumption is not valid in the real world, as specifying a random function R requires $|\mathcal{D}| \cdot \kappa$ bits (essentially one must list the value $R(x)$ for every input $x \in \mathcal{D}$), which is totally impractical given that $|\mathcal{D}|$ must be huge to ensure cryptographic security levels (e.g., $|\mathcal{D}| \geq 2^{256}$ or larger). In the real world, the random oracle is replaced with a concrete hash function like SHA-3, which is succinctly specified via, e.g., a small circuit or computer program computing the hash function. In principle, it may be possible for a cheating prover in the real world to exploit access to this succinct representation to break the security of the protocol, even if the protocol is secure in the random oracle model. However, to date no real-world protocol has been broken in this manner, and protocols that are proven secure in the random oracle model are often considered secure in practice.⁴⁷

4.7.2 The Fiat-Shamir Transformation

The Fiat-Shamir transformation [FS86] takes any public-coin IP \mathcal{I} and transforms it into a non-interactive, publicly verifiable protocol \mathcal{Q} in the random oracle model.⁴⁸ (Recall that in a public-coin protocol, the only messages from the verifier to the prover consist of the verifier’s random coin tosses).

Some approaches that do not quite work. At a high level, the Fiat-Shamir transformation mimics the transformation described in Section 3.3 that transforms any interactive proof system with a deterministic verifier into a non-interactive proof system. In that transformation, the non-interactive prover leverages the

⁴⁷The relationship between security in the random oracle model and security in the real world has been the subject of considerable debate, with a series of works identifying various protocols that are secure in the random oracle model but not secure when the random oracle is replaced with *any* concrete hash function [CGH04, BBP04, GK03, Nie02, GKMZ16]. However, these protocols and functionalities are typically contrived [KM15]. For two entertaining and diametrically opposed perspectives, the interested reader is directed to [Gol06, KM15].

⁴⁸The Fiat-Shamir transformation can also be applied to public-coin interactive argument systems to obtain non-interactive arguments in the random oracle model.

total predictability of the interactive verifier's messages to compute those messages itself on behalf of the verifier. This eliminates the need for the verifier to actually send any messages to the prover. In particular, it means that the non-interactive proof can simply specify an accepting transcript from the interactive protocol (i.e., specify the first message sent by the prover in the interactive protocol, followed by the verifier's response to that message, followed by the prover's second message, and so on until the protocol terminates).

In the setting of this section, the verifier's messages in \mathcal{I} are not predictable. But since \mathcal{I} is public coin, the verifier's messages in \mathcal{I} come from a known distribution (specifically, the uniform distribution). So a naive attempt to render the protocol non-interactive would be to ask the prover to determine the verifier's messages itself, by drawing each message at random from the uniform distribution, independent of all previous messages sent in the protocol. But this does not work because the prover is untrusted, and hence there is no way to force the prover to actually draw the verifier's challenges from the appropriate distribution.

A second approach that attempts to address the above issue is to have \mathcal{Q} use the random oracle to determine the verifier's message r_i in round i of \mathcal{I} . This will ensure that each challenge is indeed uniformly distributed. A naive attempt at implementing this second approach would be to select r_i in \mathcal{Q} by evaluating the random oracle at input i . But this attempt is also unsound. The problem is that, although this ensures each of the verifier's messages r_i are uniformly distributed, it does not ensure that they are independent of the prover's messages g_1, \dots, g_i from rounds $1, 2, \dots, i$ of \mathcal{I} . Specifically, the prover in \mathcal{Q} can learn all of the verifier's messages r_1, r_2, \dots in advance (by simply querying the random oracle at the predetermined points $1, 2, \dots$) and then choose the prover messages in \mathcal{I} in a manner that depends on these values. Since the IP \mathcal{I} is not sound if the prover knows r_i in advance of sending its i th message g_i , the resulting non-interactive argument is not sound.

The above issue can be made more concrete by imagining that \mathcal{I} is the sum-check protocol applied to an ℓ -variate polynomial g over \mathbb{F} . Consider a cheating prover \mathcal{P} who begins the protocol with a false claim C for the value $\sum_{x \in \{0,1\}^\ell} g(x)$. Suppose in round 1 of the sum-check protocol, before sending its round-1 message polynomial g_1 , \mathcal{P} knows what will be the verifier's round-1 message $r_1 \in \mathbb{F}$. Then the prover could trick the verifier as follows. If s_1 is the message that the honest prover would send in round 1, \mathcal{P} can send a polynomial g_1 such that

$$g_1(0) + g_1(1) = C \quad \text{and} \quad g_1(r_1) = s_1(r_1). \quad (4.20)$$

Note that such a polynomial is guaranteed to exist so long as g_1 is permitted to have degree at least 1. From that point on in \mathcal{I} , the cheating prover \mathcal{P} can send the same messages as the honest prover, and thereby pass all of the verifier's checks. In the naive attempt at implementing the second approach to obtaining a non-interactive protocol above, prover in \mathcal{Q} will be able to simulate this attack on \mathcal{I} . This is because the prover in \mathcal{Q} can learn r_1 by simply querying the random oracle at the input 1, and then choosing g_1 to satisfy Equation (4.20) above.

To prevent this attack on soundness, the Fiat-Shamir transformation ensures that the verifier's challenge r_i in round i of \mathcal{I} is determined by querying the random oracle at an input that depends on the prover's i 'th message g_i . This means that the prover in \mathcal{Q} can only simulate the aforementioned attack on \mathcal{I} if the prover can find a g_1 satisfying Equation (4.20) with r_1 equal to evaluation of the random oracle at the appropriate query point (which, as previously mentioned, includes g_1). Intuitively, for the prover in \mathcal{Q} to find such a g_1 , a vast number of queries to the random oracle are required, because the output of the random oracle is totally random, and for each g_1 there are a tiny number of values of r_1 satisfying Equation (4.20).

Complete description of the Fiat-Shamir transformation. The Fiat-Shamir transformation replaces each of the verifier's messages from the interactive protocol \mathcal{I} with a value derived from the random oracle in the following manner: in \mathcal{Q} , the verifier's message in round i of \mathcal{I} is determined by querying the

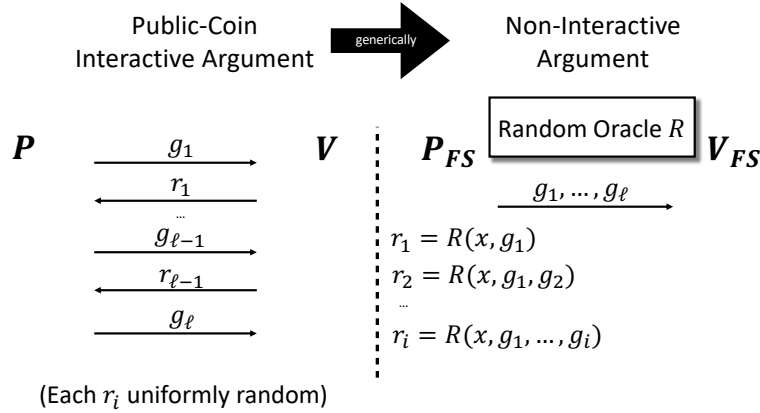


Figure 4.13: Depiction of the Fiat-Shamir transformation. Image courtesy of Ron Rothblum [Rot19].

random oracle, where the query point is the the list of messages sent by the prover in rounds $1, \dots, i - 1$.⁴⁹ As in the naive attempt above, this eliminates the need for the verifier to send any information to the prover—the prover can simply send a single message containing the transcript of the entire protocol (i.e., a list of all messages exchanged by the prover in the interactive protocol, with the verifier’s random coin tosses in the transcript replaced with the random oracle evaluations just described). See Figure 4.13.

It has long been known that when the Fiat-Shamir transformation is applied to a *constant-round* public-coin IP or argument \mathcal{I} with negligible soundness error,⁵⁰ the resulting non-interactive proof \mathcal{Q} in the random oracle model is sound against cheating provers that run in polynomial time [PS00].^{51,52} In fact, we prove this result at the end of this chapter (Theorem 4.9). However, the IPs covered in this chapter all require at least logarithmically many rounds. Recently, a better understanding of the soundness of \mathcal{Q} has been developed for such many-round protocols \mathcal{I} .

Specifically, it is now known that if a public-coin interactive proof \mathcal{I} for a language \mathcal{L} satisfies a property called *round-by-round soundness* then \mathcal{Q} is sound in the random oracle model [CCH⁺19, BCS16]. Here, \mathcal{I} satisfies round-by-round soundness if the following properties hold: (1) At any stage of any execution of \mathcal{I} , there is a well-defined state (depending on the partial transcript at that stage of the execution) and some states are “doomed”, in the sense that once the protocol \mathcal{I} is in a doomed state, it will (except with negligible probability) forever remain doomed, no matter the strategy executed by the prover in \mathcal{I} . (2) If $x \notin \mathcal{L}$, then the initial state of \mathcal{I} is doomed. (3) If at the end of the interaction the state is doomed, then the verifier will reject.⁵³

⁴⁹For the Fiat-Shamir transformation to be secure in settings where an adversary can choose the input x to the IP or argument, it is essential that x be appended to the list that is hashed in each round. This property of soundness against adversaries that can choose x is called adaptive soundness. Some real-world implementations of the Fiat-Shamir transformation have missed this detail, leading to attacks [BPW12, LPT19]. See Exercise 4.7.

⁵⁰Throughout this manuscript, negligible means any quantity smaller than the reciprocal of any fixed polynomial in the input length. Non-negligible means any quantity that is *at least* the reciprocal of some fixed polynomial in the input length.

⁵¹More quantitatively, any prover for \mathcal{Q} that can convince the verifier to accept input x with probability ϵ and runs in time T can be transformed into a prover for \mathcal{I} that convinces the verifier to accept input x with probability at least $\text{poly}(\epsilon, 1/T)$.

⁵²If \mathcal{I} is an argument rather than a proof, then soundness of \mathcal{Q} in the random oracle model will also inherit any computational hardness assumptions on which soundness of \mathcal{I} is based.

⁵³For illustration, an example of an IP with negligible soundness error that does *not* satisfy round-by-round soundness is to take any IP with soundness error $1/3$, and sequentially repeat it n times. This yields a protocol with at least n rounds and soundness error $1/3^{-\Omega(n)}$, yet it is not round-by-round sound. And indeed, applying the Fiat-Shamir transformation to this protocol does *not*

Canetti et al. [CCH⁺19] showed that the GKR protocol (and any other interactive proof based on the sum-check protocol) satisfy round-by-round soundness, and hence applying the Fiat-Shamir transformation to it yields a non-interactive proof that is secure in the random oracle model.⁵⁴

Here is some rough intuition for why round-by-round soundness of the IP \mathcal{I} implies soundness of the non-interactive proof \mathcal{Q} in the random oracle model. The only way a cheating prover in \mathcal{I} can convince the verifier of a false statement is to “get lucky”, in the sense that the verifier’s random coin tosses in \mathcal{I} happen to fall into some small set of “bad” coin tosses B that eventually force the protocol into a *non-doomed* state (B must be small because \mathcal{I} is sound). Because a random oracle is by definition totally unpredictable and unstructured, in \mathcal{Q} roughly speaking all that a cheating prover can do to find an accepting transcript is to iterate over possible prover messages/transcripts for the IP \mathcal{I} in an arbitrary order, and stop when he identifies one where the random oracle happens to return a sequence of values falling in B . Of course, this isn’t quite true: a malicious prover in \mathcal{Q} is also capable of simulating so-called *state-restoration* attacks against \mathcal{I} , which means that the prover in \mathcal{Q} can “rewind” any interaction with the verifier of \mathcal{I} to an earlier point of the interaction and “try out” sending a different response to the last message sent by the verifier in this partial transcript for \mathcal{I} . The prover may hope that by trying out a different response, this will cause the random oracle to output a non-doomed value. However, round-by-round soundness of \mathcal{I} precisely guarantees that such an attack is unlikely to succeed: once in a doomed state of \mathcal{I} , no prover strategy can “escape” doom except with negligible probability.

In summary, applying the Fiat-Shamir Transformation to a public coin interactive proof with round-by-round soundness yields a non-interactive argument in the random oracle model. The protocol can then be heuristically instantiated in the “real world” by replacing the random oracle with a cryptographic hash function (in fact, as we explain in Section 4.7.3 below, with an appropriate choice of hash function one can even prove this real-world instantiation to also be secure). The transformation itself typically adds very little concrete cost to the protocol, as the only extra work required in the real-world instantiation of the transformed protocol is that the prover and verifier have to evaluate a cryptographic hash function several times (once for each round of interaction in the original protocol). This is often a low-order cost in practice.

4.7.3 Applying the Fiat-Shamir Transformation to the GKR Protocol In the Plain Model

A recent line of work [KRR17, CCRR18, HL18, CCH⁺19, PS19] has identified a property called correlation-intractability (CI) such that the following holds. When applying the Fiat-Shamir transformation to the GKR protocol (in fact, to any IP satisfying round-by-round soundness) the resulting non-interactive protocol is a sound argument system in the *plain model* (i.e., it is sound even when the random oracle is replaced with a concrete hash function chosen at random from the CI hash family). Simplifying slightly, recent exciting progress has yielded constructions of CI hash families based on the assumed intractability of the Learning With Errors (LWE) problem, an assumption on which many lattice-based cryptosystems are based.⁵⁵ The constructions of correlation-intractable hash families have the flavor of *fully homomorphic encryption*

yield a sound argument system in the random oracle model [Rot19]. See Exercise 4.6.

⁵⁴In fact, Canetti et al. [CCH⁺19] also show that, using parallel repetition, *any* public-coin IP can be transformed into a different public-coin IP that satisfies round-by-round soundness.

⁵⁵To be precise, current constructions of hash families satisfying the precise variants of CI known to ensure plain-model soundness of the Fiat-Shamir-transformed GKR protocol require a notion called circular-secure fully homomorphic encryption [CCH⁺19]. This is a somewhat stronger assumption than intractability of LWE. Very recent work of Jawale et al. [JKKZ20] relaxes the necessary assumption to (subexponential) hardness of the LWE problem. Jawale et al. do this by introducing a variant of correlation-intractability called *lossy* correlation-intractability that also suffices to ensure soundness of the Fiat-Shamir-transformed GKR protocol in the plain model (showing this exploits additional properties of the GKR protocol beyond round-by-round soundness). They then construct lossy correlation-intractable hash families assuming subexponential hardness of LWE.

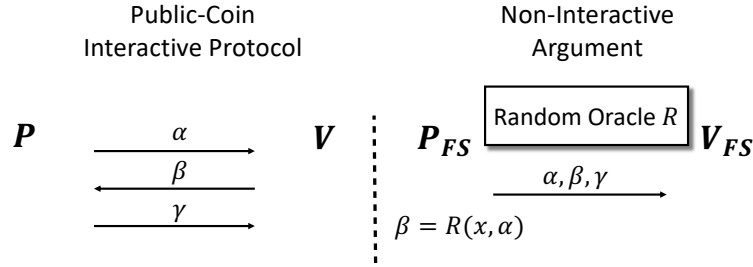


Figure 4.14: Depiction of the Fiat-Shamir transformation applied to a 3-message interactive proof or argument as in the proof of Theorem 4.9. Image courtesy of Ron Rothblum [Rot19].

(FHE) schemes, which are currently highly computationally intensive, much more so than the GKR protocol itself. However, it is plausible that cryptographic hash families used in practice actually satisfy correlation-intractability.

Fiat-Shamir and Public-Coin Arguments in the Plain Model. There is great interest in obtaining analogous results for the succinct public-coin interactive *arguments* described later in this survey, to obtain non-interactive arguments that are secure in the plain model. Unfortunately, the results that have been obtained on this topic have so far been negative. For example, Bartusek et al. [BBH⁺19] show, roughly speaking, that obtaining non-interactive arguments in this manner would require exploiting some special structure in both the underlying interactive argument and in the concrete hash function used to implement the random oracle in the Fiat-Shamir transformation.

4.7.4 Soundness in the Random Oracle Model for Constant-Round Protocols

Theorem 4.9. *Let \mathcal{I} be a constant-round public-coin IP or argument with negligible soundness error, and let \mathcal{Q} be the non-interactive protocol in the random oracle model obtained by applying the Fiat-Shamir transformation to \mathcal{I} . Then \mathcal{Q} has negligible computational soundness error. That is, no prover running in polynomial time can convince the verifier in \mathcal{Q} of a false statement with non-negligible probability.*

Proof. For simplicity, we will only prove the result in the case where \mathcal{I} is a 3-message protocol where the prover speaks first. See Figure 4.14 for a depiction of the Fiat-Shamir transformation in this setting and the notation we will use during the proof.

We will show that, for any input x , if \mathcal{P}_{FS} is a prover that runs in time T and convinces the verifier in \mathcal{Q} to accept on input x with probability at least ε (where the probability is over the choice of random oracle), then there is a prover \mathcal{P}^* for \mathcal{I} that convinces the verifier in \mathcal{I} to accept with probability at least $\varepsilon^* \geq \Omega(\varepsilon/T)$ (where the probability is over the choice of the verifier’s random challenges in \mathcal{I}). Moreover, \mathcal{P}^* has essentially the same runtime as \mathcal{P}_{FS} . The theorem follows, because if ε is non-negligible and T is polynomial in the size of the input, then ε^* is non-negligible as well.

Handling restricted \mathcal{P}_{FS} behavior. The rough idea is that there isn’t much \mathcal{P}_{FS} can do to find an accepting transcript (α, β, γ) with $\beta = R(x, \alpha)$ other than to mimic a successful prover strategy \mathcal{P} for \mathcal{Q} , setting α to be the first message sent by \mathcal{P} , setting β to be $R(x, \alpha)$, and setting γ to be \mathcal{P} ’s response to the challenge

β .⁵⁶ If this is indeed how $\mathcal{P}_{\mathcal{FS}}$ behaved, it would be easy for \mathcal{P}^* to “pull” the prover strategy \mathcal{P} for \mathcal{Q} out of $\mathcal{P}_{\mathcal{FS}}$ as follows: \mathcal{P}^* runs $\mathcal{P}_{\mathcal{FS}}$, up until the point where $\mathcal{P}_{\mathcal{FS}}$ makes (its only) query, of the form (x, α) , to the random oracle. \mathcal{P}^* sends α to the verifier \mathcal{V} in \mathcal{I} , who responds with a challenge β . \mathcal{P}^* uses β as the response of the random oracle to $\mathcal{P}_{\mathcal{FS}}$ ’s query. \mathcal{P}^* then continues running $\mathcal{P}_{\mathcal{FS}}$ until $\mathcal{P}_{\mathcal{FS}}$ terminates.

Since \mathcal{I} is public coin, \mathcal{V} chooses β uniformly random, which means that β is distributed appropriately to be treated as a response of the random oracle. Hence, with probability at least ε , $\mathcal{P}_{\mathcal{FS}}$ will produce an accepting transcript of the form (α, β, γ) . In this case, \mathcal{P}^* sends γ as its final message in \mathcal{I} , and the verifier accepts because (α, β, γ) is an accepting transcript. This ensures that \mathcal{P}^* convinces the verifier in \mathcal{I} to accept with the same probability that $\mathcal{P}_{\mathcal{FS}}$ outputs an accepting transcript, which is at least ε by assumption.

The general case: overview. In the general case, $\mathcal{P}_{\mathcal{FS}}$ may not behave in the manner above. In particular, $\mathcal{P}_{\mathcal{FS}}$ may ask the random oracle *many* queries (though no more than T of them, since $\mathcal{P}_{\mathcal{FS}}$ runs in time at most T). This means that it is not obvious which of the queries (x, α) \mathcal{P}^* should forward to \mathcal{V} as its first message α . Fortunately, we will show that it suffices for \mathcal{P}^* to simply pick one of $\mathcal{P}_{\mathcal{FS}}$ ’s queries at random. Essentially, \mathcal{P}^* will pick the “right” query with probability at least $1/T$, leading \mathcal{P}^* to convince \mathcal{V} to accept input x with probability at least ε/T .

What we can assume about $\mathcal{P}_{\mathcal{FS}}$ without loss of generality. Let us assume that $\mathcal{P}_{\mathcal{FS}}$ always makes exactly T queries to the random oracle (this can be ensured modifying $\mathcal{P}_{\mathcal{FS}}$ to ask “dummy queries” as necessary to ensure that it always makes exactly T queries to the random oracle R). Let us further assume that all queries $\mathcal{P}_{\mathcal{FS}}$ makes are distinct (there is never a reason for $\mathcal{P}_{\mathcal{FS}}$ to query the oracle at the same location twice, since the oracle will respond with the same value both times). Finally, we assume that whenever $\mathcal{P}_{\mathcal{FS}}$ successfully outputs an accepting transcript (α, β, γ) with $\beta = R(x, \alpha)$, then at least one of $\mathcal{P}_{\mathcal{FS}}$ ’s T queries to R was at point (x, α) (this can be ensured by modifying $\mathcal{P}_{\mathcal{FS}}$ to always query (x, α) before outputting the transcript (α, β, γ) , if (x, α) has not already been queried, and making a “dummy query” otherwise).

Complete description of \mathcal{P}^* . \mathcal{P}^* begins by picking a random integer $i \in \{1, \dots, T\}$. \mathcal{P}^* runs $\mathcal{P}_{\mathcal{FS}}$ up until its i ’th query to the random oracle, choosing the random oracle’s responses to queries $1, \dots, i-1$ uniformly at random. If the i th query is of the form (x, α) , \mathcal{P}^* sends α to \mathcal{V} as its first message, and receives a response β from \mathcal{V} .⁵⁷ \mathcal{P}^* uses β as the response of the random oracle to query (x, α) . \mathcal{P}^* then continues running $\mathcal{P}_{\mathcal{FS}}$, choosing the random oracle’s responses to queries $i+1, \dots, T$ uniformly at random. If $\mathcal{P}_{\mathcal{FS}}$ outputs an accepting transcript of the form (α, β, γ) , then \mathcal{P}^* sends γ to \mathcal{V} , which convinces \mathcal{V} to accept.

Analysis of success probability for \mathcal{P}^* . As in the restricted case, since \mathcal{I} is public coin, \mathcal{V} chooses β uniformly random, which means that β is distributed appropriately to be treated as a response of the random oracle. This means that $\mathcal{P}_{\mathcal{FS}}$ outputs an accepting transcript of the form $(\alpha, R(x, \alpha), \gamma)$ with probability at least ε . In this event, \mathcal{P}^* convinces \mathcal{V} to accept whenever $\mathcal{P}_{\mathcal{FS}}$ ’s i th query to R was (x, α) . Since we have assumed that $\mathcal{P}_{\mathcal{FS}}$ makes exactly T queries, all of which are distinct, and one of those queries is of the form (x, α) , this occurs with probability exactly $1/T$. Hence, \mathcal{P}^* convinces \mathcal{V} to accept with probability at least ε/T . □

⁵⁶As explained in Section 4.7.2, this isn’t actually true, as $\mathcal{P}_{\mathcal{FS}}$ can also run state-restoration attacks, an issue with which the formal proof below must grapple.

⁵⁷If the i th query is not of the form (i, α) , \mathcal{P}^* aborts, i.e., \mathcal{P}^* gives up trying to convince \mathcal{V} to accept.

Remark 4.6. Theorem 4.9 roughly shows that the Fiat-Shamir transformation renders a protocol non-interactive in the random oracle model while preserving soundness. Later in this manuscript, we will be concerned with a strengthening of soundness called *knowledge soundness* that is relevant when the prover is claiming to *know* a witness satisfying a specified property (see Section 6.3.3). Roughly speaking, the Fiat-Shamir transformation preserves knowledge soundness as well, but the analysis showing this in various contexts is more involved. See Sections 8.2.1 and 11.2.3 for details.

4.8 Exercises

Exercise 4.1. Recall that Section 4.3 gave a doubly-efficient interactive proof for counting triangles. Given as input the adjacency matrix A of a graph on n vertices, the IP views A as a function over domain $\{0, 1\}^{\log_2 n} \times \{0, 1\}^{\log_2 n}$, lets \tilde{A} denote its multilinear extension, and applies the sum-check protocol to the $(3 \log n)$ -variate polynomial

$$g(X, Y, Z) = \tilde{A}(X, Y) \cdot \tilde{A}(Y, Z) \cdot \tilde{A}(X, Z).$$

A 4-cycle in a graph is a quadruple of vertices (a, b, c, d) such that (a, b) , (b, c) , (c, d) , and (a, d) are all edges in the graph. Give a doubly-efficient interactive proof that, given as input the adjacency matrix A of a simple graph, counts the number of 4-cycles in the graph.

Exercise 4.2. Here is yet another interactive proof for counting triangles given as input the adjacency matrix A of a graph on n vertices: For a sufficiently large prime p , define $f: \{0, 1\}^{\log_2 n} \times \{0, 1\}^{\log_2 n} \times \{0, 1\}^{\log_2 n} \rightarrow \mathbb{F}_p$ via $f(i, j, k) = A_{i,j} \cdot A_{j,k} \cdot A_{i,k}$, where here we associate vectors in $\{0, 1\}^{\log_2 n}$ with numbers in $\{1, \dots, n\}$ in the natural way, and interpret entries of A as elements of \mathbb{F}_p in the natural way. Apply the sum-check protocol to the multilinear extension \tilde{f} . Explain that the protocol is complete, and has soundness error at most $(3 \log_2 n)/p$.

What are the fastest runtimes you can give for the prover and verifier in this protocol? Do you think the verifier would be interested in using this protocol?

Exercise 4.3. This question has 5 parts.

- (Part a) Section 4.2 gave a technique to take any Boolean formula $\phi: \{0, 1\}^n \rightarrow \{0, 1\}$ of size S and turn ϕ into a polynomial g over field \mathbb{F} that extends ϕ (the technique represents g via an arithmetic circuit over \mathbb{F} of size $O(S)$).

Apply this technique to the Boolean formula in Figure 4.15. You may specify the resulting extension polynomial g by drawing the arithmetic circuit computing g or by writing out some other representation of g .

- (Part b) Section 4.2 gives an interactive proof for counting the number of satisfying assignments to ϕ by applying the sum-check protocol to g . For the polynomial g you derived in Part a that extends the formula in Figure 4.15, provide the messages sent by the honest prover if the random field element chosen by the verifier in round 1 is $r_1 = 3$ and the random field element chosen by the verifier in round 2 is $r_2 = 4$. You may work over the field \mathbb{F}_{11} of integers modulo 11.
- (Part c) Imagine you are a cheating prover in the protocol of Part b above and somehow you know at the start of the protocol that in round 1 the random field element r_1 chosen by the verifier will be 3. Give a sequence of messages that you can send that will convince the verifier that the number of satisfying assignments of ϕ is 6 (the verifier should be convinced regardless of the random field elements r_2 and r_3 that will be chosen by the verifier in rounds 2 and 3).

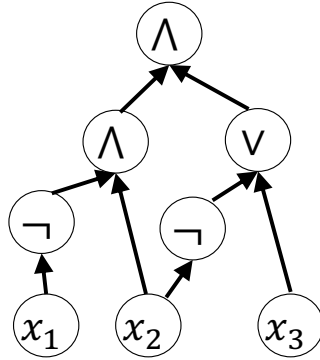


Figure 4.15: A Boolean formula ϕ over $n = 3$ variables.

- (Part d) You may notice that the extension polynomial g derived in Part a is *not* multilinear. This problem explains that there is a good reason for this.

Show that the ability to evaluate the *multilinear* extension $\tilde{\phi}$ of a formula ϕ at a randomly chosen point in \mathbb{F}^n allows one to determine whether or not ϕ is satisfiable. That is, give an efficient randomized algorithm that, given $\tilde{\phi}(\mathbf{r})$ for a randomly chosen $\mathbf{r} \in \mathbb{F}^n$, outputs SATISFIABLE with probability at least $1 - n/|\mathbb{F}|$ over the random choice of \mathbf{r} if ϕ has one or more satisfying assignments, and outputs UNSATISFIABLE with probability 1 if ϕ has no satisfying assignments. Explain why your algorithm achieves this property.

- (Part e) Let $p > 2^n$ be a prime, and as usual let \mathbb{F}_p denote the field of order p . This question establishes that the ability to evaluate $\tilde{\phi}$ at a certain specific input implies the ability not only to determine whether or not ϕ is satisfiable, but in fact to *count* the number of satisfying assignments to ϕ . Specifically, prove that

$$\sum_{x \in \{0,1\}^n} \phi(x) = 2^n \cdot \tilde{\phi}(2^{-1}, 2^{-1}, \dots, 2^{-1}).$$

Hint: Lagrange Interpolation.

Exercise 4.4. One of the more challenging notions to wrap one's head around regarding the GKR protocol is that, when applying it to a circuit \mathcal{C} with a "nice" wiring pattern, the verifier never needs to materialize the full circuit. This is because the only information about the circuit's wiring pattern of \mathcal{C} that the verifier needs to know in order to run the protocol is to evaluate $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ at a random point, for each layer i of \mathcal{C} . And $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ often have nice, simple expressions that enable them to be evaluated at any point in time logarithmic in the size of \mathcal{C} . (See Section 4.6.6).

This problem asks you to work through the details for a specific, especially simple, wiring pattern. Figures 4.5-4.8 depict (for input size $n = 4$) a circuit that squares all of its inputs, and sums the results via a binary tree of addition gates.

Recall that for a layered circuit of depth d , the layers are numbered from 0 to d where 0 corresponds to the output layer and d to the input layer.

- Assume that n is a power of 2. Give expressions $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$ for layers $i = 1, \dots, d-2$ such that the expressions can both be evaluated at any point in time $O(\log n)$ (layer i consists of 2^i addition gates, where for $j \in \{0, 1, \dots, 2^i - 1\}$, the j th addition gate has as its in-neighbors gates $2j$ and $2j+1$ at layer $i+1$).
- Assume that n is a power of two. Give expressions for $\widetilde{\text{add}}_{d-1}$ and $\widetilde{\text{mult}}_{d-1}$ that can both be evaluated at any point in time $O(\log n)$. (This layer consists of $n = 2^d$ multiplication gates, where the j th multiplication gate at layer $d-1$ has both in-neighbors equal to the j th input gate at layer d).

Exercise 4.5. Write a Python program implementing the prover and verifier in the interactive proof for counting triangles from Section 4.3 (say, over the prime field \mathbb{F}_p with $p = 2^{61} - 1$). Recall that in this interactive proof, the message from the prover in each round i is a univariate polynomial s_i of degree at most 2. To implement the prover \mathcal{P} , you may find it simplest for \mathcal{P} to specify each such polynomial via its evaluations at 3 designated inputs (say, $\{0, 1, 2\}$), rather than via its (at most) 3 coefficients. For example, if $s_i(X) = 3X^2 + 2X + 1$, it may be simplest if, rather than sending the coefficients 3, 2, and 1, the prover sends $s_i(0) = 1$, $s_i(1) = 6$ and $s_i(2) = 17$. The verifier can then evaluate $s_i(r_i)$ via Lagrange interpolation:

$$s_i(r_i) = 2^{-1} \cdot s_i(0) \cdot (r_i - 1)(r_i - 2) - s_i(1) \cdot r_i(r_i - 2) + 2^{-1} \cdot s_i(2) \cdot r_i(r_i - 1).$$

Exercise 4.6. Section 4.7.2 described the Fiat-Shamir transformation and asserted that if the Fiat-Shamir transformation is applied to any IP with negligible soundness error that satisfies an additional property called round-by-round soundness, then the resulting argument system is computationally sound in the random oracle model. One may wonder if in fact the Fiat-Shamir transformation yields a computationally sound argument for any IP with negligible soundness error, not just those that are round-by-round sound. In this problem, we will see that the answer is no.

Take any IP with perfect completeness and soundness error $1/3$, and sequentially repeat it n times, having the verifier accept if and only if all n invocations of the base IP lead to acceptance. This yields an IP with soundness error 3^{-n} . Explain why applying the Fiat-Shamir transformation to this IP does *not* yield a sound argument system in the random oracle model, despite the fact that the soundness error 3^{-n} is negligible.

You may assume that the prover in the IP is permitted to pad messages with nonces if it so desires, i.e., the prover may append extra symbols to any message and the verifier will simply ignore those symbols. For example, if the prover in the IP wishes to send message $m \in \{0, 1\}^b$, the prover could choose to send (m, m') for an arbitrary string m' , and the IP verifier will simply ignore m' .

Exercise 4.7. Recall that the GKR protocol for circuit evaluation is used to verify the claim that $\mathcal{C}(x) = y$, where \mathcal{C} is an arithmetic circuit over field \mathbb{F} , x is a vector in \mathbb{F}^n , and \mathcal{C} , x , and y are all known to both the prover and the verifier. Consider applying the Fiat-Shamir transformation in the random oracle model to the GKR protocol, but suppose that when applying the Fiat-Shamir transformation, the input x is *not* included in the partial transcripts fed into the random oracle (see Footnote 49). Show that the resulting non-interactive argument is *not* adaptively sound. That is, for a circuit \mathcal{C} and claimed output y of your choosing, explain how a cheating prover can, in time proportional to the size of \mathcal{C} and with overwhelming probability, find an input $x \in \mathbb{F}^n$ such that $\mathcal{C}(x) \neq y$, along with a convincing “proof” for the claim that $\mathcal{C}(x)$ in fact equals y .

Chapter 5

Front Ends: Turning Computer Programs Into Circuits

5.1 Introduction

In Section 4.6, we saw a very efficient interactive proof, called the GKR protocol, for verifiably outsourcing the evaluation of large arithmetic circuits, as long as the circuit is not too deep. But in the real world, people are rarely interested in evaluating giant arithmetic circuits. Rather, they typically have a computer program written in a high-level programming language like Java or Python, and want to execute the program on their data. In order for the GKR protocol to be useful in this setting, we need an efficient way to turn high-level computer programs into arithmetic circuits. We can then apply the GKR protocol (or any other interactive proof or argument system for circuit evaluation) to the resulting arithmetic circuit.

Most general purpose argument system implementations work in this two-step manner. First, a computer program is compiled into a model amenable to probabilistic checking, such as an arithmetic circuit or arithmetic circuit satisfiability instance.⁵⁸ Second, an interactive proof or argument system is applied to check that the prover correctly evaluated the circuit. In these implementations, the program-to-circuit compiler is referred to as the *front end* and the argument system used to check correct evaluation of the circuit is called the *back end*.

Some computer programs naturally lend themselves to implementation via arithmetic circuits, particularly programs that only involve addition and multiplication of integers or elements of a finite field. For example, the following layered arithmetic circuit of fan-in two implements the standard naive $O(n^3)$ time algorithm for multiplying two $n \times n$ matrices, A and B .

Let $[n] := \{1, \dots, n\}$. Adjacent to the input layer of the circuit is a layer of n^3 multiplication gates, each assigned a label $(i, j, k) \in [n] \times [n] \times [n]$. Gate (i, j, k) at this layer computes the product of $A_{i,k}$ and $B_{k,j}$. Following this layer of multiplication gates lies a binary tree of addition gates of depth $\log_2(n)$. This ensures that there are n^2 output gates, and if we assign each output gate a label $(i, j) \in [n] \times [n]$, then the (i, j) 'th output gate computes $\sum_{k \in [n]} A_{i,k} \cdot B_{k,j}$ as required by the definition of matrix multiplication. See Figure 5.1.

As another example, Figures 4.5-4.8 portray an arithmetic circuit implementing the same functionality as the computer program depicted in Algorithm 1 (with $n = 2$). The circuit devotes one layer of gates to squaring each input entry, and then sums up the results via a complete binary tree of addition gates of fan-in two.

⁵⁸Many argument systems prefer to work with models such “Rank-1 Constraint Systems” that are closely related to, but distinct from, arithmetic circuits. These alternative models are discussed later in this survey (see Section 7.5).

Algorithm 1 Algorithm Computing the Sum of Squared Entries of Input Vector

Input: Array $a = (a_1, \dots, a_n)$

- 1: $b \leftarrow 0$
- 2: **for** $i = 1, 2, \dots, n$ **do**
- 3: $b \leftarrow b + a_i^2$

Output: b

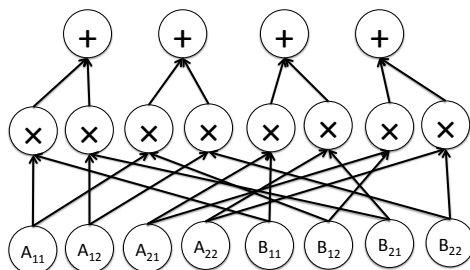


Figure 5.1: An Arithmetic Circuit Implementing The Naive Matrix Multiplication Algorithm for 2×2 matrices.

While it is fairly straightforward to turn the algorithm for naive matrix multiplication into an arithmetic circuit as above, other kinds of computer programs that perform “non-arithmetic” operations (such as evaluating complicated conditional statements) seem to be much more difficult to turn into small arithmetic circuits.

In Sections 5.3 and 5.4, we will see two techniques for turning arbitrary computer programs into circuits. In Section 5.5, we will see a third technique, which is far more practical, but makes use of what are called “non-deterministic circuits” (equivalently, the third technique produces instances of the circuit *satisfiability* problem, rather than of the circuit *evaluation* problem).

We would like to make statements like “Any computer programming that halts within T time steps can be turned into a (low-depth, layered, fan-in two) arithmetic circuit of size at most $O(T \log T)$.” In order to make statements of this form, we first have to be precise about what it means to say that a computer programs has runtime T .

5.2 Machine Code

Modern compilers are very good at efficiently turning high-level computer programs into *machine code*, which is a set of basic instructions that can each be executed in unit time on the machine’s hardware. When we say that a program runs in $T(n)$ time steps, we mean that it can be compiled into a sequence of machine instructions of length at most $T(n)$. But for this statement to be precise, we have to decide precisely what is a machine instruction. That is, we have to specify (a model of) the hardware on which we will think of our programs as running.

Our hardware model will be a simple *random access machine* (RAM). A RAM consists of the following components.

- (Main) Memory. That is, it will contain s cells of storage, where each cell can store, say, 64 bits of

data.

- A constant number (say, 8) of registers. Registers are special memory cells with which the RAM can manipulate data. That is, whereas Main Memory cells can only store data, the RAM is allowed to perform operations on data in registers, such as “add the numbers in Registers 1 and 2, and store the result in Register 3”.
- A set of $\ell = O(1)$ allowed machine instructions. Typically, these instructions are of the form:
 - Write the value currently stored in a given register to a specific location in Main Memory.
 - Read the value from a specific location in Main Memory into a register.
 - Perform basic manipulations of data in registers. For example, adding, subtracting, multiplying, dividing, or comparing the values stored in two registers, and storing the result in a third register. Or doing bitwise operations on the values stored in two registers (i.e., computing the bit-wise AND of two values). Etc.
 - A program counter. This is a special register that tells the machine what is the next instruction to execute.

5.3 A First Technique For Turning Programs Into Circuits [Sketch]

Our first technique for turning computer programs into circuits yields the following.⁵⁹ If a computer program runs in time $T(n)$ on a RAM with at most $s(n)$ cells of memory, then the program can be turned into a (layered, fan-in 2) arithmetic circuit of depth not much more than $T(n)$ and *width* of about $s(n)$ (i.e., the number of gates at each layer of the circuit is not much more than $s(n)$).

Observe that such a transformation from programs to circuits is useless in the context of the GKR protocol, because the verifier’s time complexity in the GKR protocol is at least the circuit depth, which is about $T(n)$ in this construction. In time $T(n)$, the verifier could have executed the entire program on her own, without any help from the prover. We describe this circuit-generation technique because it is conceptually important, even though it is useless in the context of the GKR protocol.

This program-to-circuit transformation makes use of the notion of a machine *configuration*. A machine configuration tells you everything about the state of a RAM at a given time. That is, it specifies the input, as well as the value of every single memory cell, register, and program counter. Observe that if a RAM has a memory of size s , then a configuration can be specified with roughly $64s$ bits (where 64 is the number of bits that can fit in one memory cell), plus some extra bits to specify the input and the values stored in the registers and program counter.

The basic idea of the transformation is to have the circuit proceed in iterations, one for each time step of the computer program. The i th iteration takes as input the configuration of the RAM after i steps of the program have been executed, and “executes one more step of the program”. That is, it determines what the configuration of the RAM would be after the $(i + 1)$ ’st machine instruction is executed. This is displayed pictorially in Figure 5.2.

A key point that makes this transformation work is that there is only a constant number of possible machine instructions, each of which is very simple (operating on only a constant number of registers, in a

⁵⁹The transformation described in this section can yield either Boolean circuits (with AND, OR, or NOT gates) or arithmetic circuits (whose inputs are elements of some finite field \mathbb{F} and whose gates compute addition and multiplication over the field). In fact, any transformation to Boolean circuits implies one to arithmetic circuits, since we know from Section 4.2 that any Boolean circuit can be transformed into an equivalent arithmetic circuit over any field, with at most a constant-factor blowup in size.

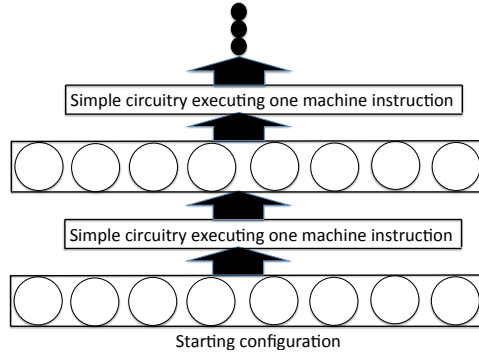


Figure 5.2: A caricature of a technique for turning any program running in time $T(n)$ and space $s(n)$ into a circuit of depth not much more than $T(n)$ and width not much more than $s(n)$.

simple manner). Hence, the circuitry that maps the configuration of the machine after i steps of the program to the configuration after the $(i + 1)$ 'st step is very simple.

Unfortunately, the circuit that are produced by this transformation have size $\tilde{\Theta}(T(n) \cdot s(n))$, meaning that, relative to running the computer program (which takes time $T(n)$), even writing down or reading the circuit is more expensive by a factor of $s(n)$.⁶⁰ The source of the inefficiency is that, for each time step of the RAM, the circuit produces an entire new machine configuration. Each configuration has size $\tilde{\Theta}(s)$, as a configuration must specify the state of the RAM's memory at that time step. Conceptually, while each step of the program only alters a constant number of memory cells in each time step, the circuit does not “know in advance” which memory cells will be updated. Hence, the circuit has to explicitly check, for each memory cell at each time step, whether or not the memory cell should be updated. This causes the circuit to be at least $s(n)$ times bigger than the runtime $T(n)$ of the RAM that the circuit simulates. This overhead renders this program-to-circuit transformation impractical.

5.4 Turning Small-Space Programs Into Shallow Circuits

The circuits that come out of the program-to-circuit transformation of Section 5.3 are useless in the context of the GKR protocol because the circuits have depth at least T . When applying the GKR protocol to such a deep circuit, the runtime of the verifier is at least T . It would be just as fast for the verifier to simply run the computer program on its own, without bothering with a prover.

A second technique for turning computer programs generates shallower circuits as long as the computer program doesn't use too much space. Specifically, it is capable of taking any program that runs in time T and space s , and turning it into a circuit of depth roughly $s \cdot \log T$ and size $2^{\Theta(s)}$.

Specifically, Section 4.5.4 explained how we can determine whether M outputs 1 on input x in less than T time steps by determining whether there is a directed path of length at most T in M 's configuration graph from the starting configuration to the accepting configuration. And to solve this problem, it suffices to

⁶⁰A second issue is that the circuit is very deep, i.e., depth at least $T(n)$. Because the GKR protocol's (Section 4.6) communication cost grows linearly with circuit depth, applying the GKR protocol to this circuit leads to communication cost at least $T(n)$, which is trivial (in the time required to read the prover's message, the verifier could afford to execute M on its own). This issue will be addressed in Section 5.4 below, which reduces the circuit depth to polynomial in the *space* usage rather than *runtime* of the RAM. However, this comes at the cost of increasing the circuit size from $\text{poly}(T, s)$ to $2^{\Theta(s)}$. Note that argument systems covered later in this manuscript do not have communication cost growing linearly with circuit depth, and hence applying these arguments to deep circuits such as those described in this section does yield non-trivial protocols.

compute a single entry of the T 'th power of A , where A be the adjacency matrix of M 's configuration graph; that is, $A_{i,j} = 1$ if configuration i has a directed edge to configuration j in M 's configuration graph, and $A_{i,j} = 0$ otherwise. This is because the (i, j) 'th entry of the T 'th power of A equals the number of directed paths of length T from node i to node j in the configuration graph of M . Hence, in order to determine whether there is a directed path of length T from the starting configuration of M to the accepting configuration, it is enough for the circuit to repeatedly square the adjacency matrix $\log_2 T$ times. We have seen in Section 5.1 that there is a circuit of size $O(N^3)$ and depth $O(\log N)$ for multiplying two $N \times N$ matrices. Since the configuration graph of M on input x is an $2^{\Theta(s)} \times 2^{\Theta(s)}$ matrix, the circuit that squares the adjacency matrix of the configuration graph of M $O(\log T)$ times has depth $O(\log(2^{\Theta(s)}) \cdot \log T) = O(s \log T)$, and size $2^{\Theta(s)}$.⁶¹

Hence, one can obtain an IP for determining the output of the RAM M by applying the GKR protocol to this circuit. However, the IP of Section 4.5.4 solves the same problem in a more direct and efficient manner.

5.5 Turning Computer Programs Into Circuit Satisfiability Instances

5.5.1 The Circuit Satisfiability Problem

In Sections 5.3 and 5.4, we saw two methods for turning computer programs into arithmetic circuits. The first method was undesirable for two reasons. First, it yielded circuits of very large depth (so large, in fact, that applying the GKR protocol to the resulting circuits led to a verifier runtime that was as bad as just having the verifier run the entire program without any help from a prover). Second, if the computer program ran in time T and space s , the circuit had size at least $T \cdot s$, and we'd really prefer to have circuits with close to T gates.

In this section, we are going to address both of these issues. However, to do so, we are going to have to shift from talking about circuit *evaluation* to talking about circuit *satisfiability*.

Recall that in the arithmetic circuit evaluation problem, the input specifies an arithmetic circuit \mathcal{C} , input x , and output(s) y , and the goal is to determine whether $\mathcal{C}(x) = y$. In the arithmetic circuit *satisfiability* problem (circuit-SAT for short), the circuit \mathcal{C} takes *two inputs*, x and w . The second input w is often called the *witness*, or sometimes the *non-deterministic input*. Given the first input x and output(s) y , the goal is to determine whether *there exists* a w such that $\mathcal{C}(x, w) = y$.

The remainder of the chapter explains that any computer program running in time T can be efficiently transformed into an instance (\mathcal{C}, x, y) of arithmetic circuit satisfiability, where the circuit \mathcal{C} has size close to T , and depth close to $\log T$. That is, the output of the program on input x equals y if and only if there exists a w such that $\mathcal{C}(x, w) = y$. Moreover, any party (such as a prover) who actually runs the program on input x can easily construct a w satisfying $\mathcal{C}(x, w) = y$.

Why Circuit-SAT instances are expressive. Intuitively, circuit satisfiability instances should be “more expressive” than circuit evaluation instances, for the same reason that *checking* a proof of a claim tends to be easier than discovering the proof in the first place. Here, the claim at hand is “running the computer program on input x yields output y ”, and, conceptually, the witness w in the circuit satisfiability instances we construct in the remainder of this chapter will represent a (traditional, static) *proof* of the claim, and the

⁶¹To be more precise, the circuit takes as input x , and first computes the adjacency matrix A of the configuration graph of M on input x (each entry of A is a simple function of x). Then the circuit repeatedly squares A to compute the $T(n)$ 'th power of A and the outputs the (i, j) 'th entry, where i indexes the starting configuration and j the ending configuration. It turns out (details omitted for brevity) that since each step of M reads and writes to $O(1)$ memory locations, the wiring predicates $\widehat{\text{add}}_i$ and $\widehat{\text{mult}}_i$ used within the GKR protocol (Section 4.6) applied to this circuit can be evaluated by the verifier in $O(ns)$ time, similar to the verifier's time cost in the IP of Section 4.5.4.

circuit \mathcal{C} will simply check that the proof is valid. Unsurprisingly, we will see that *checking validity of a proof* of this claim can be done by much smaller circuits than circuit evaluation instances that determine the veracity of the claim “from scratch”.

To make the above intuition more concrete, here is a specific, albeit somewhat contrived, example of the power of circuit satisfiability instances. Imagine a straightline program in which all inputs are elements of some finite field \mathbb{F} , and all operations are addition, multiplication, and division (by division a/b , we mean multiplying a by the multiplicative inverse of b in \mathbb{F}). Suppose one wishes to turn this straightline program into an equivalent arithmetic circuit *evaluation* instance \mathcal{C} . Since the gates of \mathcal{C} can only compute addition and multiplication operations (not division), \mathcal{C} would need to replace every division operation a/b with an explicit computation of the multiplicative inverse b^{-1} of b , where the computation of b^{-1} is only able to invoke addition and multiplication operations. This is expensive, leading to huge circuits. In contrast, to turn the straightline program into an equivalent circuit *satisfiability* instance, we can demand that the witness w contain a field element e for every division operation a/b where e should be set to b^{-1} . The circuit can “check” that $e = b^{-1}$ by adding an output gate that computes $e \cdot b - 1$. This output gate will equal 0 if and only if $e = b^{-1}$. In this manner, each division operation in the straightline program translates into only $O(1)$ additional gates and witness elements in the circuit satisfiability instance.

One may initially be worried that this technique introduces a “checker” output gate for each division operation in the straightline program, and that consequently, if there are many division operations the prover will have to send a very long message to the verifier in order to inform the verifier of the claimed output vector y of \mathcal{C} . However, since any “correct” witness w causes these “checker” gates to evaluate to 0, their claimed values are implicitly 0. This means that the size of the prover’s message specifying the claimed output vector y is independent of the number of “checker” output gates in \mathcal{C} .

5.5.2 Preview: Arguments for Circuit Satisfiability

The present chapter focuses on transforming computer programs into equivalent circuit satisfiability instances (front ends). Following this chapter, the remainder of this survey is devoted to designing succinct arguments for circuit satisfiability (back ends). As discussed in Section 5.1, the combination of front ends and back ends allows the prover to efficiently convince the verifier of the output of any specified computer program on input x .

As discussed in the previous section, one can think of the circuit satisfiability instance (\mathcal{C}, x, y) produced by the front-end as a procedure for checking the validity of a witness w of the prover’s claims. While this checking procedure is often much more efficient than the task of computing a witness from scratch, it can still be very expensive (if the witness-checking circuit \mathcal{C} is large). By applying a succinct argument to the circuit-satisfiability instance, the verifier \mathcal{V} effectively offloads the work of witness-checking onto the prover. Some succinct arguments ensure that \mathcal{V} runs in time $O(|x| + \text{polylog}(|\mathcal{C}|))$, meaning that \mathcal{V} is able to confirm the prover did the work correctly, while \mathcal{V} runs in much less time than would be required to check the witness unaided (many argument systems allow \mathcal{V} to run in less time than would be required even just to *read* the witness whose validity is being checked).

How do arguments for circuit satisfiability operate? One way we could design an efficient IP for the claim that there exists a w such that $\mathcal{C}(x, w) = y$ is to have the prover send w to the verifier, and run the GKR protocol to efficiently check that $\mathcal{C}(x, w) = y$. This would be enough to convince the verifier that indeed the program outputs y on input x . This approach works well if the witness w is small. But in the computer-program-to-circuit-satisfiability transformation that we’re about to see, the witness w will be very large (of size roughly T , the runtime of the computer program). So even asking the verifier to read the claimed witness w is as expensive as asking the verifier to simply run the program herself without the help

of a prover.

Fortunately, we will see in Section 6.3 that we can combine the GKR protocol with a cryptographic primitive called a *polynomial commitment scheme* to obtain an argument system that avoids having the prover send the entire witness w to the verifier. The high-level idea is as follows (see Section 6.3 for additional details).

In the IP for circuit satisfiability described two paragraphs above, it was essential that the prover sent the witness w at the start of the protocol, so that the prover was not able to base the choice of w on the random coin tosses of the verifier within the circuit evaluation IP to confirm that $\mathcal{C}(x, w) = y$. Put another way, the point of sending w at the start of the protocol was that it *bound* the prover to a specific choice of w *before* the prover knew anything about the verifier’s random coin tosses in the subsequent IP for circuit evaluation.

We can mimic the above, without the prover having to send w in full, using cryptographic commitment schemes. These are cryptographic protocols that have two phases: a commit phase, and a reveal phase. In a sense made precise momentarily, the commit phase binds the prover to a witness w string without requiring the prover to send w in full. In the reveal phase, the verifier asks the prover to reveal certain entries of w . The required binding property is that, unless the prover can solve some computational task that is assumed to be intractable, then after executing the commit phase, there must be some fixed string w such that the prover is forced to answer all possible reveal-phase queries in a manner consistent with w . Put another way, the prover is not able to choose w in a manner that depends on the questions asked by the verifier in the reveal phase.

This means that to obtain a succinct argument for circuit satisfiability, one can first have the prover run the commit phase of a cryptographic commitment scheme to bind itself to the witness w , then run an IP or argument for circuit evaluation to establish that $\mathcal{C}(x, w) = y$, and over the course of the protocol the verifier can force the prover as necessary to reveal any information about w that the verifier needs to know to perform its checks.

If the GKR protocol is used as the circuit evaluation protocol, what information does the verifier need to know about w to perform its checks? Recall that in order to run the GKR protocol on circuit \mathcal{C} with input $u := (x, w)$, the only information about the input that the verifier needs to know is the evaluation of the multilinear extension \tilde{u} of u at a random point, and that this evaluation is only needed by the verifier at the very end of the protocol.

We will explain in Chapter 6 that in order to quickly evaluate \tilde{u} at any point, it is enough for the verifier to know the evaluation of the multilinear extension \tilde{w} of w at a related point.

Hence, the cryptographic commitment scheme should bind the prover to the multilinear polynomial \tilde{w} , in the sense that in the reveal phase of the commitment scheme, the verifier can ask the prover to tell her $\tilde{w}(r)$ for any desired input r to w (the prover will respond with $\tilde{w}(r)$ and a small amount of “authentication information” that the verifier insists be included to enforce binding). The required binding property roughly ensures that when the verifier asks the prover to reveal $\tilde{w}(r)$, the prover will not be able to “change” its answer in a manner that depends on r .

To summarize the resulting argument system, after the prover commits to the multilinear polynomial \tilde{w} , the parties run the GKR protocol to check that $\mathcal{C}(x, w) = y$. The verifier can happily run this protocol even though it does not know w , until the very end when the verifier has to evaluate \tilde{u} at a single point (and hence at this point the verifier needs to know $\tilde{w}(r)$ for some point r). The verifier learns $\tilde{w}(r)$ from the prover, by having the prover decommit to \tilde{w} at input r .

All told, this approach (combined with the Fiat-Shamir transformation, Section 4.7.2) will lead to a non-interactive argument system of knowledge for circuit satisfiability, i.e., for the claim that the prover knows a witness w such that $\mathcal{C}(x, w) = y$. If a sufficiently efficient polynomial commitment scheme is used, the

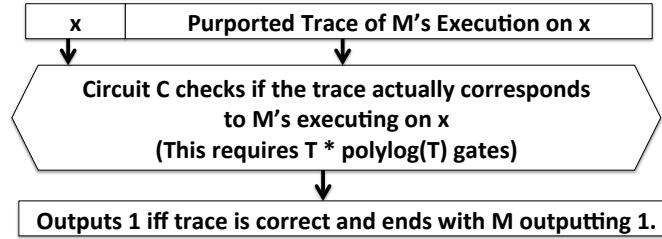


Figure 5.3: Sketch of the transformation From RAM execution on input x to an instance of circuit satisfiability.

argument system is nearly optimal in the sense that the verifier runs in linear time in the size of the input x , and the prover runs in time close to the size of \mathcal{C} .

5.5.3 The Transformation From Computer Programs To Arithmetic Circuit Satisfiability

Before describing the transformation, it is helpful to consider why the circuit generated in Method 1 of Section 5 (see Section 5.3) had at least $T \cdot s$ gates, which is significantly larger than T if s is large. The answer is that that circuit consisted of T “stages” where the i th stage computed each bit of the machine’s configuration (which includes the entire contents of its main memory) after i machine instructions had been executed.

But each machine instruction affects the value of only $O(1)$ registers and memory cells, so between any two stages, almost all bits of the configuration remain unchanged. This means that almost all of the gates and wires in the circuit are simply devoted to copying bits from the configuration after i steps to the configuration after step $i + 1$. This is highly wasteful, and in order to obtain a circuit of size close to T , rather than $T \cdot s$, we will need to cut out all of this redundancy.

To describe the main idea in the transformation, it is helpful to introduce the notion of the *transcript* (sometimes also called a *trace*) of a random access machine M 's execution on input x . Roughly speaking, the transcript describes just the *changes* to M 's configuration at each step of its execution. That is, for each step i that M takes, the transcript lists just the value of each register and the program counter at the end of step i . Since M has only $O(1)$ registers, the transcript can be specified using $O(T)$ words, where a *word* refers to a value that can be stored in a single register or memory cell.

The basic idea is that the transformation from RAM execution to circuit satisfiability produces a circuit satisfiability instance (\mathcal{C}, x, y) , where x is the input to M , y is the claimed output of M , and the witness w is supposed to be the transcript of M 's execution of input x . The circuit \mathcal{C} will simply check that w is indeed the transcript of M 's execution on input x , and if this check passes, then \mathcal{C} outputs the same value as M does according to the ending configuration in the transcript. If the check fails, \mathcal{C} outputs a special rejection symbol.

A schematic of \mathcal{C} is depicted in Figure 5.3.

5.5.4 Details of the Transformation

The circuit \mathcal{C} takes an entire transcript (sorted by time) of the entire execution of M as a non-deterministic input, where a transcript consists of (timestamp, list) pairs, one for each step taken by M . Here, a list specifies the bits contained in the current program counter and the values of all of M 's registers. The circuit then checks that the transcript is valid. This requires checking the transcript for both time consistency (i.e.,

that the claimed state of the machine at time i correctly follows from the machine’s claimed state at time $i - 1$) and memory consistency (i.e., that whenever a value is read from memory location, the value that is returned is equal to the last value written to that location).

The circuit checks time-consistency by representing the transition function of the RAM as a small sub-circuit (we provide some details of this representation later in this section). It then applies this sub-circuit to each entry i of the transcript and checks that the output is equal to entry $i + 1$ of the transcript. That is, for each time step i in $1, \dots, T - 1$, the circuit will have an output gate that will equal 0 if and only if entry $i + 1$ of the transcript equals that application of the transition function of the random access machine M to entry i of the transcript.

The circuit checks memory consistency by re-sorting the transcript based on memory location (with ties broken by time), at which point it is straightforward for the circuit to check that every memory read from a given location returns the last value written to that location. The sorting step is the most conceptually involved part of the construction of \mathcal{C} , and works as follows.

Note that all of the T time-consistency checks and all (at most) T memory-consistency checks can be done in parallel, which ensures that \mathcal{C} has polylogarithmic depth.

How to sort with a non-deterministic circuit. A *routing network* is a graph with a designated set of T source vertices and a designated set of T sink vertices (both sets of the same cardinality) satisfying the following property: for any perfect matching between sources and sinks (equivalently, for any desired sorting of the sources), there is a set of node-disjoint⁶² paths that connects each source to the sink to which it is matched. Such a set of node-disjoint paths is called a *routing*. The specific routing network used in \mathcal{C} is derived from a *De Bruijn* graph G . G consists of $\ell = O(\log T)$, with T nodes at each layer. The first layer consists of the source vertices, and the last layer consists of the sinks. Each node at intermediate layers has exactly two in-neighbors and exactly two out-neighbors.

The precise definition of the De Bruijn graph G is not essential to the discussion here. What is important is that G satisfies the following two properties.

- Property 1: Given any desired sorting of the sources, a corresponding routing can be found in $O(|G|) = O(T \cdot \log T)$ time using known routing algorithms [Ben65, Wak68, Lei92, BSCGT13a].
- Property 2: The multilinear extension of the wiring predicate of G can be evaluated in polylogarithmic time. By wiring predicate of G , we mean the Boolean function (analogous to the functions add_i and mult_i in the GKR protocol) that takes as input the labels (a, b, c) of three nodes in G , and outputs 1 if and only if b and c are the in-neighbors of a in G .

Roughly speaking, Property 2 holds because in a De Bruijn graph, the neighbors of a node with label v are obtained from v by simple bit shifts, which is a “degree-1 operation” in the following sense. The function that checks whether two binary labels are bit-shifts of each other is an AND of pairwise disjoint bit-equality checks. The direct arithmetization of such a function (replacing the AND gate with multiplication, and the bitwise equality checks with their multilinear extensions) is multilinear.

In a routing of G , each node v other than source nodes has a single in-neighbor in the routing (we think of this in-neighbor as forwarding its packet to v), and each node v other than sink nodes has exactly one out-neighbor in the routing. Thus, a routing in G can be specified by assigning each non-sink node v a single bit that specifies which of v ’s two out-neighbors in G get forwarded a packet by v . To perform the sorting

⁶²Two length- ℓ paths $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_\ell$ and $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_\ell$ are node-disjoint if there does not exist a pair $(i, j) \in [\ell] \times [\ell]$ such that $u_i = v_j$.

step, the circuit will take additional bits as non-deterministic input (i.e., as part of the witness w), called *routing bits*, which give the bit-wise specification of a routing just described.

To put everything together, the circuit \mathcal{C} sorts the (timestamps, list) pairs of the transcript from time-order into memory order by implementing the routing network G as follows. For each node v in G , \mathcal{C} contains a “gadget” of $\text{polylog}(n)$ gates. The gadget for v takes as input a (timestamp, list) pair, which is viewed as a packet, as well as the routing bit b_v for v . Based on b_v , it “forwards” its packet to the appropriate out-neighbor of v .

The Wiring Predicates of \mathcal{C} . The circuit \mathcal{C} has a very regular wiring structure, with lots of repeated structure. Specifically, its time-consistency-checking circuitry applies the same small sub-circuit (capturing the transition function of the RAM) independently to every two adjacent (timestep, list) pairs in the time-ordered transcript specified by the witness, and (after resorting the witness into memory order), the memory-consistency-checking circuitry also applied a small sub-circuit independently to adjacent (timestamp, list) pairs in the memory-ordered transcript to check that every memory read from a given location returns the last value written to that location. That is, the parts of the circuit devoted to both time-consistency and memory-consistency checking are data parallel in the sense of Section 4.6.7.

All told, it is possible to exploit this data parallel structure—and Property 2 of G above, which ensures that the sorting circuitry also has a nice, regular wiring structure—to show that (a slight modification of) the multilinear extensions add_i and mult_i of \mathcal{C} can be evaluated in polylogarithmic time.

This ensures that if one applies the GKR protocol (in combination with a commitment scheme as described in Section 5.5.2) that the verifier can run in time $O(n + \text{polylog}(T))$, without ever having to explicitly enumerate over all gates of \mathcal{C} . Moreover, the prover can generate the entire circuit \mathcal{C} and the witness w , and perform its part of the GKR protocol applied to $\mathcal{C}(x, w)$ in time $O(T \cdot \text{polylog}(T))$.

Intuitively, in the resulting argument system, the verifier is forcing the prover not only to run the RAM M on input x , but also to produce a transcript of the execution and then *confirm* via the circuit \mathcal{C} that the transcript contains no errors. Fortunately, it does not require much more work for the prover to produce the transcript and confirm its correctness than it does to run M on x in the first place.

Representing the transition function of the Random Access Machine as a small arithmetic circuit.

Depending on the field over which the circuit \mathcal{C} is defined, certain operations of the RAM are easy to compute inside \mathcal{C} using a single gate. For example, if \mathcal{C} is defined over a prime-order field \mathbb{F}_p of order p , then this field naturally simulates integer addition and multiplication so long as one is guaranteed that the values arising in the computation always lie in the range $[-\lfloor p/2 \rfloor, \lfloor p/2 \rfloor]$ (if the values grow outside of this range, then the field, by reducing all values modulo p , will no longer simulate integer arithmetic).⁶³ In contrast, fields of characteristic 2 are not able to simulate integer addition or multiplication on numbers of magnitude 2^W without spending (at least) $\Omega(W)$ gates by operating on the bit-representations of the integers. On the other hand, if \mathcal{C} is defined over a field of characteristic two, then addition of two field elements is equivalent to bitwise-XOR of the binary representations of the field operations. The message here is that integer arithmetic, but not bitwise operations, are simulated very directly over fields of large prime order (up to overflow issues), whereas bitwise operations, but not integer arithmetic, are simulated very directly over fields of characteristic 2.

In general, if a Random Access Machine has word size W then any primitive instruction other than memory accesses (e.g., integer arithmetic, bitwise operations, integer comparisons, etc.) can be implemented in a

⁶³If operating over unsigned integers rather than signed integers, the integer values arising in the computation may lie in the range $[0, p - 1]$ rather than $[-\lfloor p/2 \rfloor, \lfloor p/2 \rfloor]$.

circuit-satisfiability instance using $\text{poly}(W)$ many gates. This works by representing each bit of each register with a separate field element, and implementing the instruction bitwise. To give some simple examples, one can compute the bitwise-AND of two values $x, y \in \{0, 1\}^W$ with W multiplication gates over a large prime-order field, where the i th multiplication gate multiplies x_i by y_i . Bitwise-OR and Bitwise-XOR can be computed in a similar manner, replacing $x_i \cdot y_i$ with $x_i + y_i - x_i y_i$ and $x_i + y_i - 2x_i y_i$ respectively.

As a more complicated example, suppose the circuit is defined over a field \mathbb{F}_p of large prime order. Let a and b be two field elements interpreted as integers in $\{0, 1, \dots, p-1\}$, and suppose that one wishes to determine whether $a > b$. Let $\ell := \lceil \log_2 p \rceil$. The circuit can ensure that the witness contains 2ℓ bits $a_0, \dots, a_{\ell-1}, b_1, \dots, b_{\ell-1}$ representing the binary representations of a and b respectively as follows. First, to check that a_i is in $\{0, 1\}$ for all $i = 0, \dots, \ell-1$, the circuit can include an output gate computing $a_i^2 - a_i$. This gate will evaluate to 0 if and only if $a_i \in \{0, 1\}$. Second, to check that $(a_0, \dots, a_{\ell-1})$ is indeed the binary representation of $a \in \mathbb{F}_p$, the circuit can include an output gate computing $a - \sum_{i=0}^{\ell-1} a_i 2^i$. Assuming each $a_i \in \{0, 1\}$, this output gate equals 0 if and only if $(a_0, \dots, a_{\ell-1})$ is the binary representation of a . Analogous checks can be included in the circuit to ensure that $(b_0, \dots, b_{\ell-1})$ is the binary representation of b .

Finally, the circuit can include an output gate computing an arithmetization of the Boolean circuit that checks bit-by-bit whether $a > b$. Specifically, for $j = \ell-2, \ell-3, \dots, 1$, define

$$A_j := \prod_{j' > j} (a_{j'} b_{j'} + (1 - a_{j'})(1 - b_{j'}))$$

so that $A_j = 1$ if the $\ell - j$ high-order bits of a and b are equal. Then the following expression equals 1 if $a > b$ and 0 otherwise:

$$a_{\ell-1}(1 - b_{\ell-1}) + A_{\ell-1}a_{\ell-2}(1 - b_{\ell-2}) + \dots + A_1 \cdot a_0(1 - b_0).$$

It can be checked that the above expression (which can be computed by an arithmetic circuit of depth $O(\ell) = O(\log p)$ consisting of $O(\ell)$ gates) equals 1 if $a > b$ and 0 otherwise. Indeed, if $a_{\ell-1} = 1$ and $b_{\ell-1} = 0$, then the first term evaluates to 1 and all other terms evaluate to 0, while if $a_{\ell-1} = 0$ and $b_{\ell-1} = 1$, then all terms evaluate to 0. Otherwise, if $a_{\ell-2} = 1$ and $b_{\ell-2} = 0$, then the second term evaluates to 1 and all other terms evaluate to 0, while if $a_{\ell-2} = 0$ and $b_{\ell-2} = 1$ then all terms evaluate to 0. And so forth.

There has been considerable effort devoted to developing techniques to more efficiently simulate non-arithmetic operations over fields of large prime order. Section 5.6.3 sketches an important result in this direction, due to Bootle et al. [BCG⁺18].

5.6 Alternative Transformations and Optimizations

The previous section gave a way to turn any RAM M with runtime T into a circuit \mathcal{C} of size $\tilde{O}(T)$ such that the output of M on input x equals y if and only if there exists a w such that $\mathcal{C}(x, w) = y$. In this section, we relax this requirement on \mathcal{C} in one of two ways. First, in Section 5.6.1 we permit there to be values $y \neq M(x)$ such that there exists a w satisfying $\mathcal{C}(x, w) = y$, but we insist that if there is a polynomial-time prover capable of *finding* a w satisfying $\mathcal{C}(x, w) = y'$, then $y = M(x)$. Satisfying this requirement is still sufficient to ultimately obtain argument systems for RAM execution, by applying an argument system for circuit satisfiability to \mathcal{C} .⁶⁴ Second, in Section 5.6.2, we permit prover and verifier to interact while performing the transformation from M into \mathcal{C} (the interaction can then be removed via the Fiat-Shamir transformation).

⁶⁴More precisely, the argument system must be an argument of knowledge. See Section 6.3.3 for details.

In both of these settings, it is possible to avoid the use of routing networks in the construction of \mathcal{C} . This can be desirable because routing networks are conceptually rather complicated and can lead to noticeable concrete and asymptotic overheads (e.g., routing T items requires a routing network of size $\Omega(T \log T)$, which is superlinear in T).

5.6.1 Ensuring Memory Consistency via Merkle Trees

The point of using routing networks in \mathcal{C} was to ensure memory consistency of the execution trace specified by the witness. An alternative technique for ensuring memory consistency is to use Merkle trees, which are covered later in this survey in Section 6.3.2.2. Roughly speaking, the idea is that \mathcal{C} will insist that every memory read in the transcript is immediately followed by “authentication information” that a polynomial-time prover is only capable of producing if the value returned by the memory read is in fact the last value written to that memory location.^{65,66} This leads to a circuit \mathcal{C} such that the only *computationally tractable* method of finding a satisfying assignment w for \mathcal{C} is to provide an execution trace that indeed satisfies memory consistency. That is, while there will *exist* satisfying assignments w for \mathcal{C} that do not satisfy memory consistency, finding such assignments w would require identifying collisions in a collision-resistant family of hash functions. This technique can lead to very large circuits if there are many memory operations, because each memory operation must be followed by a full authentication path in the Merkle tree, which consists a sequence of cryptographic hash evaluations (the number of hash evaluations is logarithmic in the size of the memory). In addition to being a large number of hash values, all of which must be included in the witness w , the circuit \mathcal{C} must check that the hash evaluations in w are computed correctly, which requires the cryptographic hash function to be repeatedly evaluated inside \mathcal{C} . Cryptographic hash evaluations can require many gates to implement in an arithmetic circuit. For this and related reasons, there have been significant efforts to identify collision-resistant hash functions that are “SNARK-friendly” in the sense that they can be implemented inside arithmetic circuits using few gates [AAB⁺19, AGR⁺16, GKK⁺19, KZM⁺15b, KZM⁺15a, HBHW16, BSG120]. For machines M that perform relatively few memory operations, Merkle trees built with such SNARK-friendly hash functions can be a reasonably cost-effective technique for checking memory consistency.

5.6.2 Ensuring Memory Consistency via Fingerprinting

Another technique for checking memory consistency is to use simple fingerprinting-based memory checking techniques (recall that we discussed Reed-Solomon fingerprinting in Section 2.1). The circuit \mathcal{C} resulting from this procedure implements a *randomized* algorithm, in the following sense. In addition to public input x and witness w , \mathcal{C} takes a third input $r \in \mathbb{F}$ and the guarantee is the following: for any pair x, y ,

- if $M(x) = y$ then there exists a w such that for every $r \in \mathbb{F}$ such that $\mathcal{C}(x, w, r) = 1$. Moreover, such a w can be easily derived by any prover running M on input x .
- if $M(x) \neq y$, then for *every* w , the probability over a randomly chosen $r \in \mathbb{F}$ that $\mathcal{C}(x, w, r) = 1$ is at most $\tilde{O}(T)/|\mathbb{F}|$.

⁶⁵Each write operation must also be accompanied by authentication information to enable appropriately updating the Merkle tree. We omit details for brevity.

⁶⁶A Merkle tree is an example of a cryptographic object called an *accumulator*, which is simply a commitment to a set that furthermore supports succinct proofs of membership in the set. In this section, the relevant set is the (memory location, value) pairs comprising the RAM’s memory at a given step of the RAM’s execution during which a memory read occurs. In some applications, there can be concrete efficiency advantages to using accumulators other than Merkle trees [OWB19].

An important aspect of this transformation to be aware of is that, for any known r , it is easy for a cheating prover to find a w such that $\mathcal{C}(x, w, r) = 1$. However, if r is chosen at random from \mathbb{F} independently of w (say, because the prover commits to w and only then is public randomness used to select r), then learning that $\mathcal{C}(x, w, r) = 1$ does give very high confidence that in fact $M(x, y) = 1$. This is sufficient to combine the transformation described below with the approach described in Section 5.5.2 to obtain a succinct argument for proving that $M(x) = y$. Indeed, after the prover commits to w (or more precisely to the multilinear extension \tilde{w} of w , using a polynomial commitment scheme), public randomness can then be used to select r , and the prover can then run the GKR protocol to convince the verifier that $\mathcal{C}(x, w, r) = 1$. The resulting interactive protocol is public coin, so the interaction can be removed using the Fiat-Shamir transformation.

The idea of the randomized fingerprinting-based memory-consistency-checking procedure implemented within the circuit \mathcal{C} is the following. As we explain shortly, by tweaking the behavior of the machine M (without increasing its runtime by more than a constant factor) it is possible to ensure the following key property holds: a time-consistent transcript for M is also memory consistent if and only if the multiset of (memory location, value) pairs written from memory equals the multiset of (memory location, value) pairs read from memory. This property turns the problem of checking memory consistency into the problem of checking whether two multisets are equal—equivalently, checking whether two lists of (memory location, value) pairs are permutations of each other—and the latter can be solved with fingerprinting techniques.

We now explain how to tweak the behavior of M so as to ensure the problem of checking memory-consistency amounts to checking equality of two multisets, and then explain how to use fingerprinting to solve this latter task.

Reducing memory-consistency-checking to multiset equality checking. Here is how to tweak M to ensure that any time-consistent transcript for M is memory-consistent if and only if the multisets of (memory location, value) pairs written vs. read from memory are identical. This technique dates to work of Blum et al. [BEG⁺95] who referred to it as an *offline memory checking* procedure. At the start of the computation (time step 0), we have M initialize memory by writing an arbitrary value to each memory location, without preceding these initialization-writes with reads. After this initialization phase, suppose that we insist that every time the machine M writes a value to a memory location, it precedes the write with a read operation from the same location (the result of which is simply ignored by M), and every time M reads a value from a memory location, it follows the read with a write operation (writing the same value that was just read). Moreover, let us insist that every time a value is written to memory, M includes in the value the current timestamp. Finally, just before M terminates, it makes a linear reading scan over every memory location. Unlike all other memory reads by M , the reads during this scan are *not* followed with a matching write operation. M also halts and outputs “reject” if a read ever returns a timestamp greater than current timestamp.

With these modifications, if M does not output “reject” then the set of (memory location, value) pairs returned by all the read operations equals the set of (memory location, value) pairs written during all the write operations if and only if every write operation returns the value that was last written to that location. Clearly these tweaks only increase M ’s runtime by a constant factor, as the tweak turns each read operation and each write operation of M into both a read and a write operation.

Multiset equality checking (a.k.a. permutation checking) via fingerprinting. Recall that in Section 2.1 we gave a probabilistic procedure called Reed-Solomon fingerprinting for determining whether two vectors a and b are equal entry-by-entry: a is interpreted as specifying the coefficients of a polynomial $p_a(x) = \sum_{i=1}^n a_i x^i$ over field \mathbb{F} , and similarly for b , and the equality-checking procedure picks a random $r \in \mathbb{F}$ and checks whether $p_a(r) = p_b(r)$. The guarantee of this procedure is that if $a = b$ entry-by-entry, then the equality holds for every possible choice of r , while if a and b disagree in even a single entry i (i.e., $a_i \neq b_i$),

then with probability at least $1 - n/|\mathbb{F}|$ over the random choice of r , the equality fails to hold.

To perform memory-checking, we do not want to check equality of vectors, but rather of multisets, and this requires us to tweak to the fingerprinting procedure from Section 2.1. That is, the above reduction from memory-consistency-checking to multiset equality checking produced two lists of (memory location, value) pairs, and we need to determine whether the two lists specify the same *set* of pairs, i.e., whether they are permutations of each other; note this is different than determining whether the lists agree entry-by-entry.

To this end, let us interpret each (memory location, value) pair as a field element, via any arbitrary injection of (memory location, value) pairs to \mathbb{F} . This does require the field size $|\mathbb{F}|$ to be at least as large as the number of possible (memory location, value) pairs. For example, if the memory has size, say, 2^{64} , and values consist of 64 bits, it is sufficient for $|\mathbb{F}|$ to be at least 2^{128} . Under this interpretation, we can think of our task as follows. We are given two length- m lists of field elements $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$, where m is the number of read and write operations performed by the machine M . We want to determine whether the lists a and b are permutations of each other, i.e., whether for every possible field element $z \in \mathbb{F}$, the number of times z appears in list a equals the number of times that z appears in list b .

Here is a randomized algorithm that accomplishes this task. Interpret a as a polynomial p_a whose roots are a_1, \dots, a_m (with multiplicity), i.e., define

$$p_a(x) := \prod_{i=1}^m (a_i - x),$$

and similarly

$$p_b(x) := \prod_{i=1}^m (b_i - x).$$

Now evaluate both p_a and p_b at the same randomly chosen input $r \in \mathbb{F}$, and output 1 if and only if the evaluations are equal. Clearly p_a and p_b are the same polynomial if and only if a and b are permutations of each other. Hence, this randomized algorithm satisfies:

- if a and b are permutations of each other then this algorithm outputs 1 with probability 1.
- if a and b are not permutations of each other, then this algorithm outputs 1 with probability at most $m/|\mathbb{F}|$. This is because p_a and p_b are distinct polynomials of degree at most m and hence can agree at at most m inputs (Fact 2.1).

We can think of $p_a(r)$ and $p_b(r)$ as fingerprints of the lists a and b that captures “frequency information” about a and b (i.e., how many times each field element z appears in the two lists), but deliberately ignores the order in which A and B are presented. A key aspect of this fingerprinting procedure is that it lends itself to highly efficient implementation within arithmetic circuits. That is, given as input lists A and B of field elements, along with a field element $r \in \mathbb{F}$, an arithmetic circuit can easily evaluate $p_a(r)$ and $p_b(r)$. For example, computing $p_a(r)$ amounts to subtracting r from each input $a_i \in a$, and then computing the product of the results via a binary tree of multiplication gates. This requires only $O(m)$ gates and logarithmic depth. Hence, this randomized algorithm for permutation checking can be efficiently implemented within the arithmetic circuit \mathcal{C} .

Historical Notes and Optimizations. Techniques for memory-consistency-checking closely related to those described above were given in [ZGK⁺18] and also exploited in subsequent work [KPPS20]. Specifically, [ZGK⁺18] checks memory consistency of an execution trace for a RAM within a circuit by exploiting permutation-invariant fingerprinting to check that claimed time-ordered and memory-ordered descriptions

of the execution trace are permutations of each other. While the fingerprints can be computed within the circuit with $O(T)$ gates, this does not reduce total circuit size or prover runtime below $O(T \log T)$.⁶⁷ This holds for two reasons. First, to compute a satisfying assignment for the circuit constructed in [ZGK⁺18], the prover must to sort the transcript based on memory location, and this takes $O(T \log T)$ time. Second, there is still a need for the circuit to implement comparison operations on timestamps associated with each memory operation, and [ZGK⁺18] uses $\Theta(\log T)$ many gates to implement each comparison operation bit-wise inside the circuit-satisfiability instance (see the final paragraph of Section 5.5.4).

Both sources of overhead just described were addressed in two works [BCG⁺18, Set19]. Setty [Set19] observes that (as described above in this section), the need for the prover to sort the transcript based on memory location can be avoided by modifying the RAM as per the offline memory checking technique of Blum et al. [BEG⁺95]. This does not in general avoid the need to perform comparison operations on timestamps inside the circuit, because the modified random access machine constructed by Blum et al. [BEG⁺95] requires checking that the timestamp returned by every read operation is smaller than the timestamp at which the read operation occurs. However, there are contexts in which such comparison operations are not necessary (see, e.g., Section 13.3), and this implies $O(T)$ -sized circuits in such contexts.⁶⁸ Even outside such contexts, work of Bootle et al. [BCG⁺18] (which we sketch in Section 5.6.3 below) give a technique for reducing the *amortized* gate-complexity of performing many integer comparison operations inside a circuit over a field prime order. Specifically, they shows how to perform $O(T)$ comparison operations on integers of magnitude $\text{poly}(T)$ using $O(T)$ gates in arithmetic circuits over any prime-order field \mathbb{F}_p of size at least T .⁶⁹ In summary, both sources of “superlinearity” in the size of the memory-consistency-checking circuit and prover runtime can be removed using the techniques of [BCG⁺18, Set19], reducing both circuit size and prover runtime to $O(T)$.

Setty [Set19] and Campanelli et al. [CFQ19] observe that this fingerprinting procedure can be verified efficiently using optimized variants of succinct arguments derived from the GKR protocol [Tha13, WTS⁺18], because $p_A(r)$ can be computed via a small, low-depth arithmetic circuit with a regular wiring pattern, that simply subtracts r from each input and multiplies the results via a binary tree of multiplication gates. This ensures that the circuit-satisfiability instances resulting from the transformation above can be efficiently verified via such arguments.

Additional Applications of Fingerprinting-based Permutation Checking. The above fingerprinting procedure for checking whether two vectors are permutations of each other has a long history in algorithms and verifiable computing and has been rediscovered many times. It was introduced by Lipton [Lip89] as a hash function that is invariant to permutations of the input, and later applied in the context of interactive and non-interactive proofs with small-space streaming verifiers [Lip90, CCMT14, SLN09].

Permutation-invariant fingerprinting techniques were also applied to give zero-knowledge arguments that two encrypted vectors are permutations of each other [Nef01, GI08, Gro10b, BG12]. Such zero-knowledge arguments are also called shuffle arguments, and are directly applicable to construct an anonymous routing primitive called a mix network, a concept introduced by Chaum [Cha81]. The ideas in these works were in turn built upon to yield SNARKs for circuit satisfiability with proofs that consist of a constant number of field or group elements [GWC19, BCC⁺16, MBKM19]. Roughly speaking, these works use variants of

⁶⁷ [ZGK⁺18] asserts a prover running in time $O(T)$, but this assertion hides a factor that is linear in the word length of the RAM. [ZGK⁺18] considers this to be a constant such as 32 or 64, but in general this word length must be $\Omega(\log T)$ to write down timestamps and index into memory, if the memory has size $\Omega(T)$.

⁶⁸ Specifically, if the memory access pattern of the RAM is independent of the input, then the use of timestamps and the need to perform comparisons on them can be eliminated using a pre-processing phase requiring time $O(T)$. See Section 13.3 for details.

⁶⁹ The techniques of [BCG⁺18] build on permutation-invariant fingerprinting, and hence are interactive.

permutation checking to ensure that a purported circuit transcript assigns consistent values to all output wires of each gate, i.e., to confirm that the transcript respects the wiring pattern of the circuit. Other uses of permutation-invariant fingerprinting in the context of zero-knowledge proofs were given in [SAGL18].⁷⁰

Additional Discussion. We remark that there are other permutation-invariant fingerprinting algorithms that do *not* lend themselves to efficient implementation within arithmetic circuits, and hence are not useful for transforming an instance of RAM execution to an instance of arithmetic circuit satisfiability. An instructive example is as follows. Let \mathbb{F} be a field of prime order, and suppose that it is known that all entries of the lists a and b are positive integers with magnitude at most B , where $B \ll |\mathbb{F}|$. Then we can define the polynomial $q_a(x)$ over \mathbb{F} via

$$q_a(x) := \sum_{i=1}^m x^{a_i},$$

and similarly

$$q_b(x) := \sum_{i=1}^m x^{b_i}.$$

Clearly q_a and q_b are polynomials of degree at most B , and they satisfy properties analogous to p_a and p_b , namely:

- if a and b are permutations of each other then $q_a(r) = q_b(r)$ with probability 1 over a random choice $r \in \mathbb{F}$.
- if a and b are not permutations of each other, then $q_a(r) = q_b(r)$ with probability at most $B/|\mathbb{F}| \ll 1$. This is because q_a and q_b are distinct polynomials of degree at most B and hence can agree at at most B inputs (Fact 2.1).

However, given as input the entries of a and b , interpreted as field elements in \mathbb{F} , an arithmetic circuit cannot efficiently evaluate $q_a(r)$ or $q_b(r)$, as this would require raising r to the power of input entries, which is not a low-degree operation.

5.6.3 Efficiently Representing Non-Arithmetic Operations Over Large Prime-Order Fields

Recall from Section 5.5.4 that when operating over a field of large prime order p , it is convenient to interpret field elements as integers in $[0, p-1]$ or $[-\lfloor p/2 \rfloor, \lfloor p/2 \rfloor]$, as then integer addition and multiplication corresponds directly to field addition and multiplication, up to overflow issues. This means (again, ignoring overflow issues) integer addition and multiplication operations can be implemented with a *single* gate in the corresponding circuit satisfiability instance.

Non-arithmetic operations on integer values are more challenging to implement inside an arithmetic circuit. Section 5.5.4 described a straightforward approach, which broke field elements into their binary representation, and computed the non-arithmetic operations by operating over these bits. The reason that this bit-decomposition approach is expensive is that it transforms an integer (which for a random access machine M is a primitive data type, consuming just one machine register) into at least $\log_2 p$ field elements, and hence at least $\log_2 p$ gates. In practice, $\log_2 p$ might be roughly 128 or 256, which is a very large

⁷⁰ [SAGL18], like earlier work [CDVD⁺03], uses a collision-resistant permutation-invariant hash function to check multiset equality, rather than the simple (non-collision-resistant) permutation-invariant fingerprinting function described in this section. Such hash functions are secure against polynomial time cheating provers even when the prover knows the hash function being used in the permutation-checking procedure and can choose the inputs to the procedure.

constant. In theory, since we would like to be able to represent timestamps via a single field element, we typically think of $\log_2 p$ as at least $\Omega(\log T)$, and hence superconstant. From either perspective, turning a single machine operation such as integer comparison into (at least) 256 gates is painfully expensive.

Ideally, we would like to replace the $\Omega(\log p)$ cost of the bit-decomposition approach to implementing these operations inside a circuit with a constant independent of p . Bootle et al. [BCG⁺18] develop techniques for achieving this in an amortized sense. That is, they showed how to simulate non-arithmetic operations over integers (e.g., integer comparisons, range queries, bit-wise operations, etc.) by arithmetic circuit-satisfiability instances working over a field of large prime order. Before providing details, here is the rough idea. The bit-decomposition approach represents integers in base- b for $b = 2$, and this means that logarithmically many field elements are required to represent a single integer. The convenient feature about using base-2 was that it was easy to check that a list of field elements represented a valid base-2 representation; in particular, that every field element in the list was either 0 or 1. This is because the low-degree expression $x \mapsto x^2 - x$ equals 0 if and only if x is in $\{0, 1\}$. Instead, Bootle et al. represent integers $y \in [0, 2^W]$ in a far larger base, namely base $b = 2^{W/c}$ for some specified integer constant $c > 1$. This has the benefit that y is represented via only c field elements, rather than W field elements. However, working over such a large base b means that there is no longer a degree-2 polynomial $q(x)$ that evaluates to 0 if and only if x is in $\{0, 1, \dots, b-1\}$ —the lowest-degree polynomial q with this property has degree b . Bootle et al. work around this issue by turning the task of checking whether a field element x is in the set $\{0, 1, \dots, b-1\}$ into a *table lookup*, and then giving an efficient procedure for performing such lookups inside an arithmetic circuit satisfiability instance. That is, conceptually, they have the circuit initialize a table containing the values $\{0, 1, \dots, b-1\}$, and then have the witness include a proof that all values appearing in the base- b decomposition of any integer y arising in the computation reside in the table. As we will see, the number of gates required to initialize the table and specify and check the requisite lookup proof is roughly $\tilde{O}(b)$, so a key point is that c will be chosen to be a large enough constant so that b is smaller than the runtime T of the random access machine M whose execution the circuit is simulating. This ensures constant *amortized* cost of all the $O(T)$ decomposition operations that the circuit has to perform. Details follow.

Let 2^W be a bound on the magnitude of integers involved in each non-arithmetic operation (assume 2^W is significantly smaller than the size of the prime order field over which the circuits we generate will be defined), and let T be an upper bound on the number of operations to be simulated. In the context of Section 5.6.2, T is a bound on the runtime of the random access machine, and W is the word-size. This is because, if a register of the RAM contains W bits, then the RAM is incapable of representing integers larger than 2^W without resorting to approximate arithmetic. In this context, one would need to choose W at least as large as $\log_2 T$ to ensure that a timestamp can be stored in one machine word.

As sketched above, Bootle et al. effectively reduces each non-arithmetic operation to a lookup into a table of size $2^{W/c}$, where $c \geq 1$ is any integer parameter. For example, if $W = \ell \log_2 T$ for some constant $\ell \geq 1$, then setting $c = \ell/4$ ensures that the lookup table has size at most $T^{1/4}$. The lookup table is initialized to contain a certain set of $2^{W/c}$ pre-determined values (i.e., the values are independent of the input to the computation). In the technique of [BCG⁺18], the length of the witness w of \mathcal{C} grows linearly in c . This is because, in order to keep the table to size $2^{W/c}$, each W -bit word of memory is represented via c field elements. That is, each W -bit word is broken into c blocks of length W/c , ensuring that each block can only take on $2^{W/c}$ possible values. This means that if a transcript for a time- T computation consists of, say, $k \cdot T$ words of memory—because each time step of the transcript requires specifying k register values—the transcript will be represented by $k \cdot c \cdot T$ field elements in the witness for \mathcal{C} .

Before describing the reduction of Bootle et al. from non-arithmetic operations to lookups in a pre-determined table, we explain how to efficiently verify a long sequence of lookup operations.

Checking many lookup operations efficiently. Bootle et al. develop a technique for checking that many values all reside in the lookup table. The technique builds on the permutation-invariant fingerprinting function of Section 5.6.2. Specifically, to show that a sequence of values $\{f_1, \dots, f_N\}$ only contains elements from a lookup table containing values $\{s_1, \dots, s_B\}$ where $B \leq 2^{W/c}$ is the size of the lookup table, it is enough to show that there are non-negative integers e_1, \dots, e_B such that the polynomials $h(X) := \prod_{i=1}^N (X - f_i)$ and $q(X) := \prod_{i=1}^B (X - s_i)^{e_i}$ are the same polynomial. To establish this, the witness will specify the bit-representation of the exponents e_1, \dots, e_B (each $e_i \in \{0, 1\}^{\log_2 N}$), and the circuit confirms that $h(r) = q(r)$ for an $r \in \mathbb{F}_p$ randomly chosen by the verifier after the prover commits to the witness. As usual, Fact 2.1 implies that if this check passes then up to soundness error N/p , h and q are the same polynomial. A crucial fact that enables the circuit to efficiently implement this check is that $q(r)$ can be computed by an arithmetic circuit using $O(B \log(N))$ gates, as

$$\prod_{i=1}^B \prod_{j=1}^{\log_2 N} (r - s_i)^{2^j \cdot e_{i,j}}.$$

In summary, this lookup table technique permits Bootle et al. to implement a sequence of $O(N)$ non-arithmetic operations inside an arithmetic circuit-satisfiability instance just $O(N + B \log N)$ gates. So long as $N = O(T)$ and $B = 2^{W/c} \leq N/\log N$, this is $O(T)$ operations in total.

Reducing non-arithmetic operations to lookups. To give a sense of the main ideas of the reduction of [BCG⁺18], we sketch the reduction in the context of two specific non-arithmetic operations: range proofs and integer comparisons.

For simplicity, let us assume that $c = 2$. To confirm that a field element v is in the range $[0, 2^W]$, one can have the witness specify v 's unique representation as a pair of field elements (a, b) such that $v = 2^{W/2} \cdot a + b$ and $a, b \in \{0, \dots, 2^{W/2} - 1\}$.

The circuit then just checks that indeed $v = 2^{W/2} \cdot a + b$ and that a and b both reside in a lookup table of size $2^{W/2}$ initialized to store all field elements y between 0 and $2^{W/2} - 1$.

As another example, doing an integer comparison reduces to a range proof. Indeed, to prove that $a > c$ when a and c are guaranteed to be in $[0, 2^W]$, it is enough to show that the difference $a - c$ is positive, which is just a range proof described above, albeit under the weaker guarantee that the input $v = a - c$ to the range proof is in $[-2^W, 2^W]$ rather than $[0, 2^W]$.

5.6.4 On the Concrete Efficiency of Program-to-Circuit Transformations

The transformations from RAM execution to circuit satisfiability instances in this chapter produce circuits over fields of large prime order, with $O(T)$ gates where T is the runtime of the RAM M whose execution we wish to verify. This is clearly optimal up to a constant factor. Moreover, it is possible to ensure that the wiring of the circuit is sufficiently regular that the verifier in argument systems derived from (for example) the GKR protocol can run in time polylogarithmic in T , i.e., the verifier need not materialize the entire circuit itself.

However, these transformations can still be expensive in practice. The main reason for this is as follows.⁷¹ Section 5.5.4 describes having \mathcal{C} check an execution trace of M for time-consistency by having the circuit \mathcal{C} apply the transition function F_M of M once for each time step, to check that $F_M(s_t) = s_{t+1}$ for all $t \leq T$, where s_t denotes the state at time step t of the execution trace. Expressing the transition

⁷¹Some works include additional overheads that are substantial in practice. For example, some works (e.g., [BCG⁺13, BCTV14b]) apply the transformations of this section to a universal machine M , which takes two inputs: a computer program ϕ , and an input x , and M simulates the execution of the program ϕ on input x . The need for M to “compile” the computer program ϕ before executing ϕ on x introduces additional concrete overheads. See Section 16.2 for additional discussion.

function of M as an arithmetic sub-circuit can easily require thousands of gates, which means that the circuit \mathcal{C} will have size *at least* several thousands of times larger than the runtime of M . Roughly speaking, this overhead can be attributed to the fact that, at each time step, M executes only one instruction, yet the transition function of M must capture “all possible instructions” that might be executed at any time step. This approach, of repeating the entire transition function of M within \mathcal{C} for each time step $t = 1, \dots, T$, is implemented in some works [BCG⁺13, BCTV14b, BSBC⁺17, BBHR19]. Yet this overhead can be avoided entirely for computations where the precise instruction executed at a given time-step does not depend on the input. A quintessential example is naive matrix multiplication (see the circuit in Figure 5.1), as the operation performed at each time-step of this algorithm is entirely input-oblivious.

Several papers develop alternative front ends that aim to address the overheads described above, which can be several orders of magnitude (see for example [BFR⁺13, WSR⁺15, ZGK⁺18]). Unfortunately, there is currently no implemented front end that is able to leverage input-oblivious behavior in computer programs to produce smaller circuits, while simultaneously guaranteeing the production of circuits that have a sufficiently regular wiring structure to avoid the need for the verifier to materialize the full circuit.

5.7 Exercises

Exercise 5.1. Describe a layered arithmetic circuit of fan-in three that takes as input a matrix $A \in \{0, 1\}^{n \times n}$, interprets A as the adjacency matrix of a graph G , and outputs the number of triangles in G . You may assume that n is a power of three.

Exercise 5.2. Describe a layered arithmetic circuit of fan-in two that, given as input an $n \times n$ matrix A with entries from some field \mathbb{F} , computes $\sum_{i,j,k,\ell \in \{1,\dots,n\}} A_{i,j} \cdot A_{k,\ell}$. The smaller your circuit is, the better.

Exercise 5.3. Fix an integer $k > 0$. Assume that k is a power 2 and let $p > k$ be a large prime number. Describe an arithmetic circuit of fan-in 2 that takes as input n elements of the field \mathbb{F}_p , a_1, a_2, \dots, a_n , and outputs the n field elements $a_1^k, a_2^k, \dots, a_n^k$.

What is the verifier’s asymptotic runtime when the GKR protocol is applied to this circuit (express your answer in terms of k and n)? Would the verifier be interested in using this protocol if n is very small (say, if $n = 1$)? What if n is very large?

Exercise 5.4. Let $p > 2$ be prime. Draw an arithmetic circuit \mathcal{C} over \mathbb{F}_p that takes as input one field element $b \in \mathbb{F}_p$ and evaluates to 0 if and only if $b \in \{0, 1\}$.

Exercise 5.5. Let $p = 11$. Draw an arithmetic circuit \mathcal{C} over \mathbb{F}_p that takes as input one field element a followed by four field elements b_0, b_1, b_2, b_3 , and such that all output gates of \mathcal{C} evaluate to 0 if and only if (b_0, b_1, b_2, b_3) is the binary representation of a . That is, $b_i \in \{0, 1\}$ for $i = 1, \dots, 4$, and $a = \sum_{i=0}^3 b_i \cdot 2^i$.

Exercise 5.6. Let $p = 11$. Let $x = (a, b)$ consist of two elements of the field \mathbb{F}_p . Draw an arithmetic circuit satisfiability instance that is equivalent to the conditional $a \geq b$. That is, interpreting a and b as integers in $\{0, 1, \dots, p-1\}$, the following two properties should hold:

- $a \geq b \implies$ there exists a witness w such that evaluating \mathcal{C} on input (x, w) produces the all-zeros output.
- $a < b \implies$ there does not exist a witness w such that evaluating \mathcal{C} on input (x, w) produces the all-zeros output.

Additional Exercises. The interested reader can find a sequence of additional exercises on front ends at <https://www.pepper-project.org/tutorials/t3-biu-mw.pdf>. These exercises discuss transforming computer programs into equivalent R1CS-satisfiability instances, a generalization of arithmetic circuit-satisfiability that we discuss further in Section 7.5.

Chapter 6

A First Succinct Argument for Circuit Satisfiability, from Interactive Proofs

In Section 5.5 we saw an efficient way to turn any computer program into an equivalent instance of the arithmetic circuit satisfiability problem. Roughly, we showed that the problem of checking whether a random access machine M taking at most T steps on an input of size x produces output y can be reduced to a circuit satisfiability instance (C, x, y) , where C has size close to T and depth close to $O(\log T)$. That is, M outputs y on x if and only if there exists a w such that $C(x, w) = y$.

This transformation is only useful in the context of interactive proofs and arguments if we can design efficient proof systems for solving instances of circuit satisfiability. In this section, we will see our first example of such an argument system, by combining the GKR protocol with a cryptographic primitive called a *polynomial commitment scheme*, in a manner that was already outlined in detail in Section 5.5.2. The polynomial commitment scheme we describe in this chapter is conceptually appealing but highly impractical; more practical polynomial commitment schemes will be covered later in this manuscript.

6.1 A Naive Approach: An IP for Circuit Satisfiability

A naive way to use the GKR protocol to solve circuit satisfiability is to have the prover explicitly send to the verifier the witness w satisfying $C(x, w) = y$, and then running the GKR protocol to check that indeed $C(x, w) = y$. The problem with this simple approach is that in many settings w can be very large. For example, in the transformation from Section 5.5, the witness w is supposed to be a transcript of M 's entire execution on input x , and hence w has size at least T . This means that in the time that the verifier would take to read the whole witness, the verifier could have run M on x without any help from the prover.

6.2 Succinct Arguments for Circuit Satisfiability

If an argument system for circuit satisfiability avoids the above bottleneck of sending the entire witness to the verifier, then it is called *succinct*. Formally, we say that an argument system for circuit satisfiability is succinct if the total communication is sublinear in the size of the witness $|w|$.⁷² Succinctness is important

⁷²Here, sublinear in $|w|$ means $o(|w|)$, i.e., any expression that asymptotically is much smaller than the witness length. This use of the term “succinct” is slightly nonstandard, as many works reserve the term *succinct* for any proof or argument systems in which the total communication is polylogarithmic (or even logarithmic) in the witness length (or even in the circuit size). Some others use succinctness more informally to refer broadly to argument systems with short proofs.

for a variety of reasons:

- Shorter proofs are always better. For example, in applications to block-chains, proofs must be stored in the block chain permanently. If proofs are long, it drastically increases the global storage requirements of the block chain.
- In some applications, witnesses are naturally large. For example, consider a hospital that publishes cryptographic hash $h(w)$ of a massive database w of patient records, and later wants to prove that it ran a specific analysis on w . In this case, the witness is the database w , the public input x is the hash value $h(w)$, and the circuit \mathcal{C} should both implement the analysis of w and “check” that $h(w) = x$.
- Known transformations from computer programs to circuit satisfiability produce circuits with very large witnesses (see Section 5.5).

The coming chapters will describe a variety of approaches to obtaining succinct arguments.⁷³ This section will cover one specific approach.

6.3 A First Succinct Argument for Circuit Satisfiability

6.3.1 The Approach

The approach of this section is to “simulate” the trivial application of the GKR protocol to circuit satisfiability described in Section 6.1, but *without* requiring the prover to explicitly send w to the verifier. We will accomplish this by using a cryptographic primitive called a *polynomial commitment scheme*. The idea of combining the GKR protocol with polynomial commitment schemes to obtain succinct arguments was first put forth by Zhang et al. [ZGK⁺17a]. We cover polynomial commitment schemes with state-of-the-art concrete efficiency in Chapter 13. In this section, we informally introduce the notion of polynomial commitment schemes and sketch a conceptually simple (but impractical) polynomial commitment scheme based on low-degree tests and Merkle trees.

Cryptographic Commitment Schemes. Conceptually, cryptographic commitment schemes can be described via the following metaphor. They allow the committer to take some object b (b could be a field element, vector, polynomial, etc.) place b in a box and lock it, and then send the locked box to a “verifier”. The committer holds on to the key to the lock. Later, the verifier can ask the committer to open the box, which the committer can do by sending the verifier the key to the lock. Most commitment schemes satisfy two properties: hiding and binding. In the metaphor, hiding means that the verifier can not “see inside” the locked box to learn anything about the object within it. Binding means that once the box is locked and transmitted to the verifier, the committer cannot change the object within the box. We provide a far more detailed and formal treatment of cryptographic commitment schemes much later in this survey, in Section 11.3.

Polynomial Commitment Schemes. Roughly speaking, a polynomial commitment scheme is simply a commitment scheme in which the object being committed to is (all evaluations of) a low-degree polynomial.

⁷³There is strong evidence that succinct interactive *proofs* (as opposed to arguments) for circuit satisfiability do not exist [BHZ87, PSSV07, GVW02, Wee05]. For example, it is known [GVW02] that interactive proofs for circuit satisfiability cannot have communication cost that is *logarithmic* in the witness length unless $\mathbf{coNP} \subseteq \mathbf{AM}$, and this is widely believed to be false (i.e., it is not believed that there are efficient constant-round interactive proofs to establish that a circuit is *not* satisfiable). Similar (albeit quantitatively weaker) surprising consequences are known to follow from the existence of interactive proofs for circuit satisfiability with sublinear (rather than logarithmic) communication cost.

That is, a polynomial commitment scheme allows a prover to commit to a low-degree polynomial \tilde{w} and later reveal $\tilde{w}(r)$ for a point r of the verifier's choosing. Even though in the commitment phase the prover does *not* send all evaluations of \tilde{w} to the verifier, the commitment still effectively binds the prover to a specific \tilde{w} . That is, at a later time, the verifier can ask the prover to reveal $\tilde{w}(r)$ for any desired r of the verifier's choosing, and the prover is effectively forced to reveal $\tilde{w}(r)$ for a fixed polynomial \tilde{w} determined at the time of the original commitment. In particular, the prover is unable to choose the polynomial \tilde{w} to depend on the query point r , at least not without breaking the computational assumption on which security of the commitment scheme is based.

Combining Polynomial Commitment Schemes and the GKR Protocol. When applying the GKR protocol to check that $\mathcal{C}(x, w) = y$, the verifier does not need to know any information whatsoever about w until the very end of the protocol, when (as explained in Section 6.3.2.1 below) the verifier needs to know $\tilde{w}(r)$ for a randomly chosen input r .

So rather than having the prover send w in full to the verifier as in Section 6.1, we can have the prover merely send a *commitment* to \tilde{w} at the start of the protocol. The prover and verifier can then happily apply the GKR protocol to the claim that $\mathcal{C}(x, w) = y$, ignoring the commitment entirely until the very end of the protocol. At this point, the verifier needs to know $\tilde{w}(r)$. The verifier can force the prover to reveal this quantity using the commitment protocol.

Because the polynomial commitment scheme *bound* the prover to a fixed multilinear polynomial \tilde{w} , the soundness analysis of the argument system is essentially the same as if the prover had sent all of w explicitly to the verifier at the start of the protocol as in Section 6.1 (see Section 6.3.3 for additional details of how one formally analyzes the soundness of this argument system).

6.3.2 Details

6.3.2.1 What The GKR Verifier Needs to Know About The Witness

In this subsection, we justify the assertion from Section 6.3.1 that the only information the verifier needs about w in order to apply the GKR protocol to check that $\mathcal{C}(x, w) = y$ is $\tilde{w}(r_1, \dots, r_{\log n})$.

Let z denote the concatenation of x and w . Let us assume for simplicity throughout this section that x and w are both of length n , so that each entry of z can be assigned a unique label in $\{0, 1\}^{1+\log n}$, with the i th entry of x assigned label $(0, i)$, and the i th entry of w assigned label $(1, i)$.

A key observation is that when applying the GKR protocol to check that $\mathcal{C}(z) = y$, the verifier doesn't need to know the exact value of z . Rather, the verifier only needs to know $\tilde{z}(r_0, \dots, r_{\log n})$ at a single, randomly chosen input $(r_0, \dots, r_{\log n})$. Moreover, the verifier doesn't even need to know $\tilde{z}(r)$ until *the very end of the protocol*, after the interactive with the prover has finished. We now explain that in order to calculate $\tilde{z}(r)$, it suffices for the verifier to know $\tilde{w}(r_1, \dots, r_{\log n})$.

It is straightforward to check that

$$\tilde{z}(r_0, r_1, \dots, r_{\log n}) = (1 - r_0) \cdot \tilde{x}(r_1, \dots, r_{\log n}) + r_0 \cdot \tilde{w}(r_1, \dots, r_{\log n}). \quad (6.1)$$

Indeed, the right hand side is a multilinear polynomial in $(r_0, r_1, \dots, r_{\log n})$ that evaluates to $z(r_0, \dots, r_{\log n})$ whenever $(r_0, \dots, r_{\log n}) \in \{0, 1\}^{1+\log n}$.⁷⁴ By Fact 3.5, the right hand side of Equation (6.1) must equal the unique multilinear extension of z .

⁷⁴To see that the latter statement holds, observe that the right hand side of Equation (6.1) evaluates to $\tilde{x}(r_1, \dots, r_{\log n})$ when $r_0 = 0$ and to $\tilde{w}(r_1, \dots, r_{\log n})$ and when $r_0 = 1$. Since \tilde{x} and \tilde{w} extend x and w respectively, this means that the right hand side extends the concatenated input (x, w) .

Equation (6.1) implies that, given $\tilde{w}(r_1, \dots, r_{\log n})$, the verifier can evaluate $\tilde{z}(r_0, \dots, r_{\log n})$ in $O(n)$ time, since the verifier can evaluate $\tilde{x}(r_1, \dots, r_{\log n})$ in $O(n)$ time (see Lemma 3.8).

In summary, the GKR protocol has the (a priori) amazing property that in order for the verifier to apply it to a known circuit \mathcal{C} on input $z = (x, w) \in \mathbb{F}^n \times \mathbb{F}^n$, the verifier does not need to know anything at all about w other than a *single* field element, namely a single evaluation of \tilde{w} . Moreover, the verifier doesn't even need to know this single field element until the very end of the protocol, after the entire interaction with the prover has terminated.

6.3.2.2 A First Polynomial Commitment Scheme

There are a number of ways to design polynomial commitment schemes. In this section, we describe a simple, folklore polynomial commitment scheme (it was also explicitly proposed by Yael Kalai [Kal17]).⁷⁵ This scheme is impractical owing to a large prover runtime (see paragraph on “Costs of this argument system” later in this section), but it provides a clean and simple introduction to cryptographic commitment schemes. We will see (much) more efficient examples of polynomial commitment schemes later in the survey.⁷⁶

The scheme makes essential use of two important concepts: Merkle Trees and low-degree tests.

Merkle Trees. A Merkle tree [Mer79] (sometimes also called a hash tree) can be used to design a *string-commitment scheme*, which allows a sender to send a short commitment to a *string* $s \in \Sigma^n$ for any finite alphabet Σ . Later, the sender can efficiently reveal the value of any entries of s that are requested by the receiver.

Specifically, a Merkle tree makes use of a collision-resistant hash function h mapping inputs to $\{0, 1\}^\kappa$ where κ is a security parameter that in practice is typically on the order of several hundred.^{77,78}

The leaves of the tree are hashes of data blocks (i.e., each leaf is a symbol of a string s), and every internal node of the tree is assigned the hash of its two children. Figure 6.1 provides a visual depiction of a hash tree.

One obtains a string-commitment protocol from a Merkle tree as follows. In the commitment step, the sender commits to the string s by sending the root of the hash-tree.

⁷⁵The polynomial commitment scheme described here, through its use of a low-degree test, only binds the prover to a function that is *close* to a low-degree polynomial. This issue complicates the soundness analyses of argument systems that make use of such proofs of *proximity* to low-degree polynomials. We discuss ways to address this issue later in this section and in Sections 7.2.1.4 and 9.2.2. In contrast, the polynomial commitment schemes that we cover later in Chapter 13 in fact bind the prover to an actual low-degree polynomial.

⁷⁶In particular, practical argument systems have replaced low-degree tests with interactive variants that have far superior concrete efficiency (see Sections 9.4 and 9.6); the interaction can then be removed from the argument via the Fiat-Shamir transformation. This is analogous to how probabilistically checkable proofs such as the one described in Section 8.4 have been replaced in practice with interactive variants (e.g., Section 9.3). For this reason, we do not cover low-degree tests or their analysis in detail in this survey.

⁷⁷For example, SHA-3 allows for several output sizes, from as small as 224 bits to as large 512 bits.

⁷⁸As discussed in Section 4.7.2, cryptographic hash functions such as SHA-3 or BLAKE3 are designed with the goal of ensuring that they “behave like” truly random functions. In particular, for such cryptographic hash functions it is typically assumed that the fastest way to find a collision is via exhaustive search, i.e., randomly choosing inputs at which to evaluate the hash function until a collision is found. If the hash function were a truly random function mapping to range $\{0, 1\}^\kappa$, then by the *birthday paradox*, with high probability roughly $\sqrt{2^\kappa} = 2^{\kappa/2}$ evaluations must be performed before exhaustive search finds a collision. This means that for security against attackers running in time, say, 2^{128} , the output size of the hash function should consist of at least $\kappa = 256$ bits. Obtaining security against attackers running in time 2^λ is often referred to by saying the primitive “achieves λ bits of security”, and λ is called the *security parameter*. Quantum algorithms are in principle capable of finding collisions in random functions in time $2^{\lambda/3}$ via a combination of Grover’s algorithm and random sampling [BHT98], meaning that λ should be set larger by a factor of $\frac{3}{2}$ to achieve security against the same number of quantum rather than classical operations.

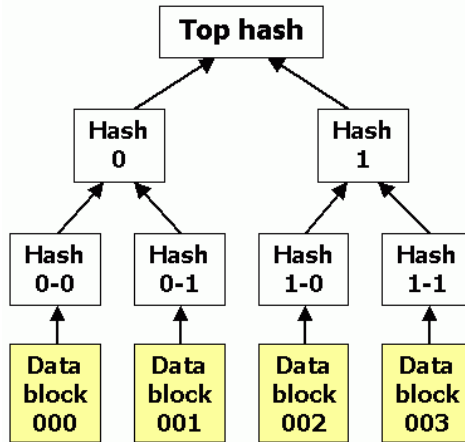


Figure 6.1: A hash tree. Image source: http://commons.wikimedia.org/wiki/File:Hash_tree.png

If the sender is later asked to reveal the i th symbol in s , the sender sends the value of the i th leaf in the tree (i.e., s_i), as well as the value of every node v along the root-to-leaf path for s_i , and the sibling of each such node v . We call all of this information the *authentication information* for s_i . The receiver checks that the hash of every two siblings sent equals the claimed value of their parent.

Since the tree has depth $O(\log n)$, this translates to the sender sending $O(\log n)$ hash values per symbol of s that is revealed.

The scheme is binding in the following sense. For each index i , there is at most one value s_i that the sender can successfully reveal without finding a collision under the hash function h . This is because, if the sender is able to send valid authentication information for two different values s_i and s'_i , then there must be at least one collision under h along the root-to-leaf path connecting the root to the i th leaf, since the authentication information for both s_i and s'_i result in the same root hash value, but differ in at least one leaf hash value.

A polynomial commitment scheme from a Merkle tree? One could attempt to obtain a polynomial commitment scheme directly from a Merkle tree by having the prover Merkle-commit to the string consisting of all evaluations of the polynomial, i.e., $p(z_1), \dots, p(z_N)$ where z_1, \dots, z_N is an enumeration of all possible inputs to the polynomial. This would enable the prover to reveal any requested evaluation of the polynomial: if the verifier asks for $p(z_i)$, the prover can reply with $p(z_i)$ along authentication information for this value (the authentication information consists of $O(\log N)$ hash values).

Unfortunately, this approach does *not* directly yield a polynomial commitment scheme. The reason is that while the Merkle tree does bind the prover to a fixed string, the string could be totally unstructured and in particular may not be close to the list of evaluations of any low-degree polynomial. Put another way, using a Merkle tree in this manner will bind the prover to some *function* (i.e., after the commit phase, there is a fixed function p such that the prover must answer any query z_i in the evaluation phase with $p(z_i)$), but there is no guarantee that p is a low-degree polynomial.

To address this issue, we combine Merkle trees with a low-degree test. The low-degree test ensures that not only is the prover bound to some (possibly completely unstructured) string, but actually that the string contains all evaluations of a low-degree polynomial (more precisely, it ensures that the string is “close” to the evaluation-table of a low-degree polynomial). The low-degree guarantees this despite only inspecting

a small number of entries of the string—often logarithmic in the length of the string—thereby keeping the amount of authentication information transmitted by the prover low (at least, lower than the communication that would be required to explicitly send a complete description of the polynomial to the verifier). Details follow.

Low-Degree Tests. Suppose a receiver is given oracle access to a giant string s , which is claimed to contain all evaluations of an m -variate function over a finite field \mathbb{F} (note that there are $|\mathbb{F}|^m$ such inputs, so s consists of a list of $|\mathbb{F}|^m$ elements of \mathbb{F}). A low-degree test allows one determine whether or not the string is consistent with a low-degree polynomial, by looking at only a tiny fraction of symbols within the string.

Unfortunately, because the low-degree test only looks at a tiny fraction of s , it cannot determine whether s is *exactly* consistent with a low-degree polynomial (imagine if s were obtained from a low-degree polynomial p by changing its value on only one input. Then unless the test gets lucky and chooses the input on which s and p disagree, the test has no hope of distinguishing between s and p itself).⁷⁹

What the low-degree test can guarantee, however, is that s is *close* in Hamming distance to (the string of all evaluations of) a low-degree polynomial. That is, if the test passes with probability γ , then there is a low-degree polynomial that agrees with s on close to a γ fraction of points.

Typically, low-degree tests are extremely simple procedures, but they are often very complicated to analyze (and existing analyses often involve very large constants that result in weak guarantees unless the field size is very large). An example of such a low-degree test is the point-versus-line test of Rubinfeld and Sudan, with a tighter analysis subsequently given by Arora and Sudan [AS03]. In this test, one evaluates s along a randomly chosen line in \mathbb{F}^m , and confirms that s restricted to this line is consistent with a univariate polynomial of degree at most m . Clearly, if the string s agrees perfectly with a multilinear then this test will always pass. The works [RS96, AS03] roughly show that if the test passes with probability γ , then there is a low-degree polynomial that agrees with s at close to a γ fraction of points (in this survey, we will not discuss how these results are proved).⁸⁰

A Polynomial Commitment Scheme by Combining Merkle Trees and Low-Degree Tests. Let $\tilde{w}: \mathbb{F}^{\log n} \rightarrow \mathbb{F}$ be a $(\log n)$ -variate multilinear polynomial over \mathbb{F} . Let s be the string consisting of all $|\mathbb{F}|^{\log n}$ evaluations of \tilde{w} . One obtains a polynomial commitment scheme by applying the Merkle-tree based string commitment scheme of Section 6.3.2.2, and then applying a low-degree test to s (i.e., if the point-versus-line low-degree test is used, then the receiver picks a random line in $\mathbb{F}^{\log n}$, asks the sender to provide authentication information for all points along the line, and checks that the revealed values are consistent with a univariate polynomial of degree at most $\log n$).

The guarantee of this polynomial commitment scheme is the same as in the string-commitment scheme of Section 6.3.2.2, except that the use of the low-degree test ensures that if the sender passes all of the receivers checks with probability γ , then not only is the sender bound to a fixed string s , but also that there is some low-degree polynomial that agrees with s at close to a γ fraction of points.

This guarantee is enough to use the polynomial commitment scheme in conjunction with the GKR protocol applied to the claim $\mathcal{C}(x, w) = y$, as outlined in Section 6.3.1. Specifically, if the verifier’s checks in the polynomial commitment scheme pass with probability at least (say) $1/2$, then the prover is bound to

⁷⁹The word “test” in the phrase low-degree test has precise technical connotations. Specifically, it refers to the fact that if a function passes the test, then the function is only guaranteed to be “close” to a low-degree polynomial, i.e., it may not be exactly equal to a low-degree polynomial. This is the same sense that the word test is used in the field of property testing (see https://en.wikipedia.org/wiki/Property_testing). We reserve the word “test” throughout this manuscript to have this technical connotation.

⁸⁰More precisely, these works show that there is a low-degree polynomial that agrees with s on at least a $\gamma - m^{O(1)}/|\mathbb{F}|^{\Omega(1)}$ fraction of points. This fraction is $\gamma - o(1)$ as long as $|\mathbb{F}|$ is super-polynomially large in m (or even a large enough polynomial in m).

a string s such that there is a multilinear polynomial p that agrees with s on close to a $1/2$ fraction of points. As long as the point $(r_1, \dots, r_{\log n})$ at which the verifier in the GKR protocol evaluates s is not one of the “bad” points on which s and p disagree, then the soundness analysis of the GKR protocol applies exactly as if the prover were bound to the multilinear polynomial p itself.

This is enough to argue that if the prover passes all of the verifier’s checks with probability significantly larger than $1/2$, then indeed there exists a w (namely, the restriction of p to the domain $\{0, 1\}^{\log n}$) such that $\mathcal{C}(x, w) = y$. The soundness error can be reduced from roughly $1/2$ to arbitrarily close to 0 by repeating the protocol many times and rejecting if any of the executions ever results in a rejection.⁸¹

Costs of this succinct argument system. In addition to the communication involved in applying the GKR protocol to check that $\mathcal{C}(x, w) = y$, the argument system above requires additional communication for the prover to commit to \tilde{w} and execute the point-versus-line low-degree test. The total communication cost due to the polynomial commit scheme is $O(|\mathbb{F}| \cdot \log n)$ hash values (the cost is dominated by the cost of the prover revealing the value of \tilde{w} on all $|\mathbb{F}|$ points along a line chosen by the verifier). This is $O(n)$ hash values as long as $|\mathbb{F}| \leq n/\log n$ (note that, while in practice we prefer to work over large fields, the soundness error of the GKR protocol is $O\left(\frac{d \log |\mathcal{C}|}{|\mathbb{F}|}\right)$, so working over a field of size $O(n/\log n)$ is enough to ensure non-trivial soundness error in the GKR protocol as long as $d \log |\mathcal{C}| \ll n/\log n$.)

The verifier’s runtime is the same as in the GKR protocol, plus the time required to play its part of the polynomial commit scheme. Assuming the collision-resistant hash function h can be evaluated in constant time, and the field size is $O(n/\log^2 n)$, the verifier spends $O(n)$ time to execute its part of the polynomial commitment scheme.

The prover’s runtime in the above argument system is dominated by the time required to commit to \tilde{w} . This requires building a Merkle tree over all possible evaluations of \tilde{w} , of which there are $|\mathbb{F}|^{\log n}$. If we work over a field of size (say) $O(n)$, then this runtime is $n^{O(\log n)}$, which is superpolynomial. So, as described, this polynomial commitment scheme is asymptotically efficient for the verifier, but not the prover.

Remark 6.1. It is possible to reduce the prover’s runtime to $O(n^c)$ for some constant c in the above argument system. The way to do this is to tweak the parameters within the GKR protocol to enable working over a much smaller field, of size $O(\text{polylog}(n))$. This will be explained in more detail in Section 8.3 when we talk about designing succinct arguments from PCPs and multi-prover interactive proofs. However, the resulting prover runtime will still be impractical (practicality requires a prover runtime close to *linear*, rather than polynomial, in the size of the circuit-satisfiability instance). As indicated above, working over such a small field would also lead to soundness error of $1/\text{polylog}(n)$, and the protocol would have to be repeated many times to drive the soundness error low enough for cryptographic use.

Remark 6.2. An alternative polynomial commitment scheme would be to use Merkle trees to have the prover commit to the string consisting of the n coefficients of the multilinear polynomial $\tilde{w}: \mathbb{F}^{\log n}$, rather than to the $|\mathbb{F}|^{\log n}$ evaluations of \tilde{w} . This approach would have the benefit of allowing the commitment to be computed with $O(n)$ cryptographic hash evaluations, and the commitment would remain small (consisting simply of the root hash evaluation). However, in order to reveal the evaluation $\tilde{w}(r)$ for a point $r \in \mathbb{F}^n$, the prover would have to reveal all n of the coefficients of \tilde{w} , resulting in linear communication complexity and verifier runtime. This is no more efficient than the naive interactive proof from Section 6.1 in which \mathcal{P} simply sends w to \mathcal{V} at the start of the protocol (and the naive approach has the benefit of being statistically rather than computationally sound).

⁸¹It is possible to use a so-called *list-decoding guarantee* of the low-degree test to argue that the soundness error is much lower than $1/2$ (if the field size is large enough), without the need for repetition of the protocol. See Section 7.2.1.4 for details.

6.3.3 Knowledge Soundness

Proofs and Arguments of Knowledge. The notion of a proof or argument of knowledge is meant to capture the situation where a prover establishes not only that a statement is true, but also that the prover *knows* a “witness” w to the validity of the statement. For example, in the authentication application of Chapter 1, Alice chooses a password x at random, publishes the hash value $y = h(x)$ of x , and later Alice wants to prove to a verifier that she *knows* a preimage of y under h , i.e., a w such that $h(w) = y$.

What does it mean for Alice to prove that she *knows* a preimage of y under h ? The notion of proof-of-knowledge (also called *knowledge soundness*) posits the following answer. If Alice convinces a verifier to accept her proof with non-negligible probability, then there should be a polynomial time algorithm \mathcal{E} that, if given the ability to repeatedly interact with Alice, is able to output a preimage w of y under h with non-negligible probability. \mathcal{E} is called an *extractor* algorithm. The idea of this definition is that, since \mathcal{E} is efficient, it can’t know anything more than Alice does (i.e., anything \mathcal{E} can compute efficiently by interacting with Alice, Alice could compute efficiently on her own, by simulating \mathcal{E} ’s interaction with herself). Hence, since \mathcal{E} can efficiently find w by interacting with Alice, then Alice must know w . One may think of \mathcal{E} as “efficiently pulling w out of Alice’s head”.⁸²

The argument system for arithmetic circuit satisfiability in this section (obtained by combining the GKR interactive proof with a commitment c to the multilinear polynomial \tilde{w}) is in fact an argument of knowledge. The rough idea is to assume that the polynomial commitment scheme satisfies a property called extractability, which is a stronger property than mere binding. Roughly speaking, extractability guarantees that there is efficient extractor algorithm \mathcal{E}' satisfying the following. If an efficient prover is capable of passing all of the checks performed in the commit and reveal phase of the polynomial commitment scheme with non-negligible probability, then \mathcal{E}' can actually extract from the prover a low-degree polynomial p that is consistent with any values revealed by the prover in the reveal phase of the commitment scheme. The polynomial commitment scheme described in Section 6.3.2.2 is extractable (we justify this assertion in Section 8.2.1).

The extractability guarantee of the polynomial commitment scheme enables one to take efficient prover \mathcal{P}^* for the argument system that convinces the argument system verifier to accept with non-negligible probability ϵ , and extract from \mathcal{P}^* a witness w and prover strategy \mathcal{P} that convinces the verifier within the GKR protocol that $\mathcal{C}(x, w) = y$. Details follow.

Recap of the succinct argument system. Recall that the argument-system prover \mathcal{P}^* first sends a commitment c to a multilinear polynomial p claimed to extend a witness w such that $\mathcal{C}(x, w) = y$. After receiving c , the argument system verifier \mathcal{V}^* acts identically to the GKR verifier \mathcal{V} , i.e., \mathcal{V}^* simulates \mathcal{V} and copies its behavior (\mathcal{V}^* can do this despite not knowing w , because the GKR verifier \mathcal{V} does not need to know anything about w until the very end of the GKR protocol). Similarly, the honest argument-system prover \mathcal{P}^* acts identically to the honest GKR prover for the claim that $\mathcal{C}(x, w) = y$.

At the very end of the GKR protocol, the GKR verifier \mathcal{V} being simulated by \mathcal{V}^* does need to evaluate the multilinear extension \tilde{w} of w at a random point r in order to make its final accept/reject decision. \mathcal{V}^* obtains this evaluation using the evaluation procedure of the polynomial commitment scheme applied to the commitment c , and outputs whatever accept/reject decision \mathcal{V} would output given the evaluation $p(r)$ obtained from the commitment scheme.

Knowledge soundness of the argument. Now suppose that \mathcal{P}^* is a (polynomial time, but possibly malicious) argument-system prover strategy that convinces the argument-system verifier \mathcal{V}^* to accept with some

⁸²The interested reader is directed to [Gol07, Section 4.7] for a detailed discussion of how to formalize knowledge soundness.

non-negligible probability ε . To establish knowledge soundness of the argument system, we need to explain that there is an efficient extraction procedure \mathcal{E} that can pull out of \mathcal{P}^* a witness w^* such that $\mathcal{C}(x, w^*) = y$.

The extractability of the polynomial commitment scheme implies that we can efficiently extract a pre-image p of the commitment c sent by \mathcal{P}^* at the start of the argument, i.e., p is a multilinear polynomial that opens to all the same values as c . \mathcal{E} sets w^* to be the witness that p extends, i.e., w^* is the set of all evaluation of p at inputs in the Boolean hypercube, $\{0, 1\}^{\log|w|}$.

We still need to explain that w^* satisfies $\mathcal{C}(x, w^*) = y$. To do this, we construct a GKR prover strategy \mathcal{P} that convinces the GKR verifier \mathcal{V} to accept the claim that $\mathcal{C}(x, w^*) = y$ with probability ε . The soundness of the GKR protocol then implies that indeed $\mathcal{C}(x, w^*) = y$.

\mathcal{P} simply simulates \mathcal{P}^* starting from right after \mathcal{P}^* sent the commitment c . That is, in every round i of the GKR protocol, \mathcal{P} sends to \mathcal{V} the message m_i that \mathcal{P}^* would send in that round of the argument system. The GKR verifier \mathcal{V} will reply to m_i with a response r_i , and \mathcal{P} then continues simulating \mathcal{P}^* into the next round, using r_i as the response of the argument-system verifier \mathcal{V}^* to m_i .

By construction, \mathcal{P} convinces the GKR verifier \mathcal{V} to accept the claim that $\mathcal{C}(x, w^*) = y$ with exactly the same probability epsilon that \mathcal{P}^* convinces the argument-system verifier \mathcal{V}^* to accept, namely ε . This concludes the proof.

Because the succinct argument of this section is in fact a public coin argument of knowledge, combining it with the Fiat-Shamir transformation yields our first succinct non-interactive argument of knowledge, or SNARK. This SNARK is unconditionally secure in the random oracle model.

Chapter 7

MIPs and Succinct Arguments

While MIPs are of some interest in their own right, we will see later that they can be building blocks for constructing succinct arguments. In particular, at the end of Section 7.2, we outline how to obtain a succinct argument from a state-of-the-art MIP.⁸³ Succinct arguments based on this approach are arguably the conceptually simplest in the literature, and implementations have achieved state-of-the-art performance [Set19]. MIPs are also of significant historical importance, and the state-of-the-art MIP in Section 7.2 exhibits several ideas that will recur in more complicated forms later in this manuscript.

7.1 MIPs: Definitions and Basic Results

Definition 7.1. A k -prover interactive proof protocol for a language $\mathcal{L} \subseteq \{0,1\}^*$ involves $k+1$ parties: a probabilistic polynomial time verifier, and k provers. The verifier exchanges a sequence of messages with each prover; each prover's message is a function of the input and the messages from \mathcal{V} that it has seen so far. The interaction produces a transcript $t = (\mathcal{V}(r), \mathcal{P}_1, \dots, \mathcal{P}_k)(x)$, where r denotes \mathcal{V} 's internal randomness. After the transcript t is produced, \mathcal{V} decides whether to output accept or reject based on r, t , and x . Denote by $\text{out}(\mathcal{V}, x, r, \mathcal{P}_1, \dots, \mathcal{P}_k)$ the output of verifier \mathcal{V} on input x given prover strategies $(\mathcal{P}_1, \dots, \mathcal{P}_k)$ and that \mathcal{V} 's internal randomness is equal to r .

The multi-prover interactive proof system has completeness error δ_c and soundness error δ_s if the following two properties hold.

1. (*Completeness*) There exists a tuple of prover strategies $(\mathcal{P}_1, \dots, \mathcal{P}_k)$ such that for every $x \in \mathcal{L}$,

$$\Pr[\text{out}(\mathcal{V}, x, r, \mathcal{P}_1, \dots, \mathcal{P}_k) = \text{accept}] \geq 1 - \delta_c.$$

2. (*Soundness*) For every $x \notin \mathcal{L}$ and every tuple of prover strategies $(\mathcal{P}'_1, \dots, \mathcal{P}'_k)$,

$$\Pr[\text{out}(\mathcal{V}, x, r, \mathcal{P}'_1, \dots, \mathcal{P}'_k) = \text{accept}] \leq \delta_s.$$

Say that a k -prover interactive proof system is valid if $\delta_c, \delta_s \leq 1/3$. The complexity class **MIP** is the class of all languages possessing valid k -prover interactive proof systems, for some $k = \text{poly}(n)$.

⁸³When an initial version of this manuscript was publicly released in the form of lecture notes in 2018, this approach to obtaining succinct arguments had not been previously published; the only published approach to turning MIPs into succinct arguments at that time [BC12] made use of a cryptographic primitive known as fully homomorphic encryption, which is currently much too computationally intensive to yield practical SNARKs. Since that time, Setty [Set19] has implemented and extended the MIP-to-succinct-argument approach described in this manuscript.

The MIP model was introduced by Ben-Or, Goldwasser, Kilian, and Wigderson [BGKW88]. It is crucial in Definition 7.1 that each prover’s message is a function only of the input and the messages from \mathcal{V} that it has seen so far. In particular, \mathcal{P}_i cannot tell \mathcal{P}_j what messages \mathcal{V} has sent it, or vice versa, for any $i \neq j$. If such “cross-talk” between \mathcal{P}_i and \mathcal{P}_j were allowed, then it would be possible to simulate any MIP by a single-prover interactive proof, and the classes **MIP** and **IP** would become equal.

As discussed in Section 1.2.3, it can be helpful to think of MIP as follows. The provers are like prisoners who are about to be interrogated. The prisoners get placed in separate interrogation rooms. Prior to going into these rooms, the prisoners can talk amongst themselves, plotting a strategy for answering questions. But once they are placed in the rooms, they can no longer talk to each other, and in particular prover i cannot tell the other provers what questions the verifier is asking it. The verifier is like the interrogator, trying to determine if the prover’s stories are consistent with each other, and with the claim being asserted.

The next section shows that, up to polynomial blowups in \mathcal{V} ’s runtime, 2-prover MIPs are just as expressive as k -prover MIPs, for any $k = \text{poly}(n)$.

7.1.1 What Does a Second Prover Buy?

Non-Adaptivity. In a single-prover interactive proof, the prover \mathcal{P} is allowed to act adaptively, in the sense that \mathcal{P} ’s response to the i th message m_i sent from \mathcal{V} is allowed to depend on the preceding $i - 1$ messages. Intuitively, the reason that MIPs are more expressive than IPs is that the presence of a second prover (who does not know \mathcal{V} ’s messages to the first prover) prevents the first prover from behaving in this adaptive manner.⁸⁴ This can be formalized via the following easy lemma showing that the complexity class **MIP** is equivalent to the class of languages satisfied by polynomial time randomized *oracle machines*. Here, an oracle machine is essentially a computer that has query access to a giant string \mathcal{O} that is fixed at the start of the computer’s execution. The string \mathcal{O} may be enormous, but the computer is allowed to look at any desired symbol \mathcal{O}_i (i.e., the i th symbol of \mathcal{O}) in unit time. One can think of any query that the computer makes to \mathcal{O} as a question, and \mathcal{O}_i as the answer. Because \mathcal{O} is fixed at the start of the computer’s execution, the answers that are returned by \mathcal{O} are non-adaptive in the sense that the answer to the computer’s j th question *does not depend on which questions the computer asked previously*.

Lemma 7.2 ([FRS88]). *Let \mathcal{L} be a language, and M a probabilistic polynomial time oracle Turing Machine such that $x \in \mathcal{L} \implies \exists$ an oracle \mathcal{O} such that $M^{\mathcal{O}}$ accepts x with probability 1, and $x \notin \mathcal{L} \implies \forall$ oracles \mathcal{O} , $M^{\mathcal{O}}$ rejects x with probability at least $2/3$. Then there is a (2-prover) MIP for \mathcal{L} .*

Remark 7.1. In Lemma 7.2, one can think of \mathcal{O} as a giant purported proof that $x \in \mathcal{L}$, and machine M only looks at a small (i.e., polynomial) number of symbols of the proof. This is the same notion as a *probabilistically checkable proof*, which we will introduce formally in Section 8.1. In this terminology, Lemma 7.2 states that any PCP with a polynomial time verifier can be turned into a 2-prover MIP with a polynomial time verifier.

Proof. \mathcal{V} simulates M , and every time M poses a query q to the oracle, \mathcal{V} asks the query to \mathcal{P}_1 , treating \mathcal{P}_1 ’s response as $\mathcal{O}(q)$. At the end of the protocol, \mathcal{V} picks a query q uniformly at random from all queries that were posed to \mathcal{P}_1 , and poses it to \mathcal{P}_2 , rejecting if \mathcal{P}_2 ’s response to q does not equal \mathcal{P}_1 ’s. Finally, the protocol is repeatedly independently 3ℓ times, where ℓ is (an upper bound on) the number of queries that M

⁸⁴One may initially have the intuition that, since allowing adaptivity on the part of the prover means allowing “more expressive” prover strategies, prover adaptivity leads to efficient proof systems for more challenging problems. In fact, the opposite is true. Allowing the prover to behave adaptively gives the prover more power to break soundness. Hence, allowing the prover to behave adaptively actually weakens the class of problems that have proof systems with an efficient verifier.

poses to the oracle on any input $x \in \{0, 1\}^n$ (note that ℓ is at most polynomial in the input size n , since M runs in polynomial time). \mathcal{V} accepts only if all instances accept.

Completeness is clear: if $x \in \mathcal{L}$, there is some oracle \mathcal{O}^* causing M to accept x with probability 1. If \mathcal{P}_1 and \mathcal{P}_2 respond to any query q with $\mathcal{O}^*(q)$, then \mathcal{V} will accept x on each of the runs of the protocol with probability 1.

For soundness, observe that since \mathcal{P}_2 is only asked a single query, we can treat \mathcal{P}_2 as an oracle \mathcal{O} . That is, \mathcal{P}_2 's answer on query q is a function only of q . On any run of the protocol, let q_1, \dots, q_ℓ denote the queries that \mathcal{V} poses to \mathcal{P}_1 on input x . On the one hand, if \mathcal{P}_1 ever answers a query q_i differently than $\mathcal{O}(q_i)$, the verifier will pick that query to pose to \mathcal{P}_2 and reject, with probability at least $1/\ell$. On the other hand, if \mathcal{P}_1 answers every query q_i with $\mathcal{O}(q_i)$, then \mathcal{V} will reject with probability at least $2/3$ because $M^{\mathcal{O}}$ rejects with that probability. Therefore, \mathcal{V} rejects on each run of the protocol with probability at least $1/\ell$, and hence \mathcal{V} rejects on at least one run of the protocol with probability at least $1 - (1 - 1/\ell)^{3\ell} > 2/3$. \square

The same argument implies that any k -prover MIP (with completeness error at most $\delta_c \leq 1/(9\ell)$, where ℓ is the total number of queries asked) can be simulated by a 2-prover MIP [BGKW88]. In the simulation, \mathcal{V} poses all of the questions from the k -prover MIP to \mathcal{P}_1 , then picks a question at random and poses it to \mathcal{P}_2 , rejecting if the answers do not agree. \mathcal{P}_2 can be treated as an oracle since \mathcal{P}_2 is only posed a single question (and hence has no opportunity to behave adaptively), and if \mathcal{P}_1 answers even a single query q_i “non-adaptively” (i.e., different than how \mathcal{P}_2 would answer), the probability this is detected is at least $1/\ell$. The whole 2-prover protocol must be repeated $\Omega(\ell)$ times to drive the soundness error from $1/\ell$ down to $1/3$.

In summary, one can both force non-adaptivity and reduce the number of provers to 2 by posing all queries to \mathcal{P}_1 and choosing one of the queries at random to pose to \mathcal{P}_2 . While this conveys much of the intuition for why MIPs are more expressive than IPs, the technique is very expensive in practice, due to the need for $\Omega(\ell)$ repetitions (typically, ℓ is on the order of $\log n$, and can easily be in the hundreds in implementations). Fortunately, the MIP that we describe in Section 7.2 requires only two provers without the need for repetition to force non-adaptivity or reduce the number of provers to 2.

But What Does Non-Adaptivity Buy? We will see in Section 7.2 that non-adaptivity buys *succinctness for NP statements*. That is, we will give a MIP for arithmetic circuit *satisfiability* (as opposed to circuit evaluation) in which the total communication and verifier runtime is sublinear in the size of the witness w .

This should not be surprising, as we saw the same phenomenon in Chapter 6. There, we used a polynomial commitment scheme to cryptographically *bind* the prover to a multilinear polynomial \tilde{w} that was fixed at the start of the interaction with the verifier. In particular, the polynomial commitment scheme enforced non-adaptivity, i.e., the prover must tell the verifier $\tilde{w}(r)$, and is not able to “change its answer” based on the interaction with the verifier. The addition of a second prover in a 2-prover MIP has exactly the same effect. Indeed, we will see that the second prover in the MIP of Section 7.2 essentially functions as a polynomial commitment scheme; see Section 7.3.

7.2 An Efficient MIP For Circuit Satisfiability

Warmup: A 2-Prover MIP for Low-Depth Arithmetic Circuit Satisfiability. The succinct argument from Chapter 6 can be directly adapted to yield a 2-prover MIP. The idea is to use the second prover to function as the polynomial commitment scheme.

In more detail, the verifier uses the first prover to apply the GKR protocol to the claim $\mathcal{C}(x, w) = y$. As explained in Section 6.3.2.1, at the end of this protocol, the prover makes a claim about $\tilde{w}(r)$.

In Chapter 6, this claim was checked by forcing the prover to reveal $\tilde{w}(r)$ via the polynomial commitment protocol (which itself involved a set-commitment combined with a low-degree test).

In the MIP, the verifier simply uses the second prover to play the role of the polynomial commitment scheme (i.e., to provide a claimed value for $\tilde{w}(r)$ that does not depend on the questions the verifier asked to the first prover, and to execute a low-degree test).

For example, if the low-degree test used is the point-versus-line test, then the verifier picks a random line λ in $\mathbb{F}^{\log n}$ containing r , and sends λ to the second prover, who is asked to respond with a univariate polynomial of degree $\log n$ claimed to equal \tilde{w} restricted to λ . Since r is on the line λ , this univariate polynomial implicitly specifies $\tilde{w}(r)$, and the verifier checks that this value matches the first prover's claim about $\tilde{w}(r)$.

A downside of the warm-up 2-prover MIP for arithmetic circuit satisfiability is that the communication cost and the verifier's runtime grow linearly with the circuit depth. Hence, the protocol does not save the verifier time for deep, narrow circuits. In general, this is not a major downside, because Section 5.5 explained that *any* computer program running in time T can be turned into an equivalent instance of arithmetic circuit satisfiability where the circuit is short and wide rather than long and narrow (specifically, the circuit has depth roughly $O(\log T)$ and size $\tilde{O}(T)$). Nonetheless, it is interesting to give a 2-prover MIP that directly handles deep, narrow circuits as efficiently as it handles short and wide ones. In so doing, we will see some ideas that will recur in later sections when we study argument systems based on PCPs and linear PCPs.

The 2-Prover MIP described in the remainder of this section is a refinement of one given by Blumberg et al. [BTWV14]. It combines several new ideas with techniques from the original $\text{MIP} = \text{NEXP}$ proof of [BFL91], as well as the GKR protocol [GKR08] and its refinements by Cormode, Mitzenmacher, and Thaler [CMT12].

7.2.1 Protocol Summary

7.2.1.1 Terminology

Let \mathcal{C} be an arithmetic circuit over a field \mathbb{F} taking an explicit input x and a non-deterministic input w . Let $S = 2^k$ denote the number of gates in \mathcal{C} , and assign each gate in \mathcal{C} a binary label in $\{0, 1\}^k$. Refer to an assignment of values to each gate of \mathcal{C} as a *transcript* of \mathcal{C} , and view the transcript as a function $W : \{0, 1\}^k \rightarrow \mathbb{F}$ mapping gate labels to their values.

Given a claim that $\mathcal{C}(x, w) = y$, a *correct transcript* is a transcript in which the values assigned to the input gates are those of x , the intermediate values correspond to the correct operation of each gate in \mathcal{C} , and the values assigned to the output gates are y . The arithmetic circuit satisfiability problem on instance Given a triple $\{\mathcal{C}, x, y\}$ is equivalent to determining whether there is a correct transcript for $\{\mathcal{C}, x, y\}$.

7.2.1.2 The MIP

The MIP works by having \mathcal{P}_1 claim find and “hold” an extension Z of a correct transcript W for $\{\mathcal{C}, x, y\}$. If the prover is honest, then Z will equal \tilde{W} , the multilinear extension of W . The protocol then identifies a polynomial $g_{x,y,Z} : \mathbb{F}^{3k} \rightarrow \mathbb{F}$ (which depends on x , y , and Z) satisfying the following property: $g_{x,y,Z}(a, b, c) = 0$ for all Boolean inputs $(a, b, c) \in \{0, 1\}^{3k} \iff Z$ is indeed an extension of a correct transcript W .

To check that $g_{x,y,Z}$ vanishes at all Boolean inputs, the protocol identifies a related polynomial $h_{x,y,Z}$ such that $g_{x,y,Z}$ vanishes at all Boolean inputs \iff the following equation holds:

$$\sum_{(a,b,c) \in \{0,1\}^{3k}} h_{x,y,Z}(a, b, c) = 0. \quad (7.1)$$

(Strictly speaking, the polynomial $h_{x,y,Z}$ is randomly generated, and there is a small chance over the random choice of $h_{x,y,Z}$ that Equation (7.1) holds even though $g_{x,y,Z}$ does not vanish at all Boolean inputs). The MIP applies the sum-check protocol to the polynomial $h_{x,y,Z}$ to compute this sum (note that if Z is a low-degree polynomial, then so is $h_{x,y,Z}$, as is required both to control costs and guarantee soundness in the sum-check protocol).

At the end of the sum-check protocol, \mathcal{V} needs to evaluate $h_{x,y,Z}$ at a random point, which in turn requires evaluating Z at a random point $r \in \mathbb{F}^k$. Unfortunately, \mathcal{V} cannot compute $Z(r)$, since \mathcal{V} does not have access to the polynomial Z (as Z only “exists” in \mathcal{P}_1 ’s head). Instead, \mathcal{V} asks \mathcal{P}_2 to send her $Z(r)$, using a primitive called a *low-degree test* (we discussed low-degree tests in Section 6.3.2.2). Specifically, \mathcal{P}_2 is asked to send Z restricted to a plane Q , where Q is chosen to be a random plane in \mathbb{F}^k containing r . This forces \mathcal{P}_2 to implicitly make a claim about $Z(r)$ (note that \mathcal{P}_2 does not know *which* point in Q is r); \mathcal{V} rejects if \mathcal{P}_1 and \mathcal{P}_2 ’s claims about $Z(r)$ are inconsistent, and accepts otherwise.

The low-degree test cannot guarantee that Z itself is a low-degree polynomial, since \mathcal{V} only ever inspects Z at a small number of points. Hence it is impossible to argue that $h_{x,y,Z}$ itself satisfies Equation (7.1): the soundness analysis for the sum-check protocol breaks down if the polynomial to which it is applied has large degree. However, the low-degree test *does* guarantee that if \mathcal{P}_1 and \mathcal{P}_2 ’s claims about $Z(r)$ are consistent with non-negligible probability over the random choice of r , then Z is close to a low-degree polynomial Y , in the sense that $Y(r') = Z(r')$ for a large fraction of points $r' \in \mathbb{F}^{3k}$. Since $h_{x,y,Y}$ is low-degree, it is straightforward to tweak the soundness analysis of the sum-check protocol to argue that $h_{x,y,Y}$ satisfies Equation (7.1), and hence that Y extends a correct transcript for $\{\mathcal{C}, x, y\}$ (cf. Theorem 7.4).

Preview: The importance of checking that a polynomial vanishes on designated a subspace. The problem of checking that a certain polynomial $g_{x,y,Z}$ vanishes on a designated subspace plays a central role in many MIPs and PCPs. The problem is sometimes referred to as checking a *Vanishing Reed-Solomon code* [BS08]. This problem will arise several more times in this survey, including in state of the art PCPs, IOPs, and linear PCPs described several chapters hence. One difference is that in the PCPs, IOPs, and linear PCPs of later sections, the polynomial $g_{x,y,Z}$ is *univariate*, instead of $(3 \log S)$ -variate as in the MIP considered here.

Comparison to the GKR Protocol. While the GKR protocol verifies the claim $\mathcal{C}(x, w) = y$ layer by layer, with a different instance of the sum-check protocol required for each layer of \mathcal{C} , the MIP of this section verifies the whole circuit in one shot, using a single invocation of the sum-check protocol. The reason the GKR protocol must work layer-by-layer is that the verifier must force the prover to make a claim about (the multilinear extension of) *the input alone*, since the verifier never materializes the intermediate gates of the circuit. This is not necessary in the multi-prover setting: in the MIP, \mathcal{P}_1 makes a claim about an extension Z of *the entire transcript*. \mathcal{V} cannot check this claim independently, but that is okay because there is a second prover to ask for help.

7.2.1.3 Protocol Details

Notation. Let $\text{add}, \text{mult}: \{0, 1\}^{3k} \rightarrow \{0, 1\}$ denote the functions that take as input three gate labels (a, b, c) from \mathcal{C} and outputs 1 if and only if gate a adds (respectively, multiplies) the outputs of gates b and c . While the GKR protocol had separate functions add_i and mult_i for each layer of \mathcal{C} , the MIP of this section arithmetizes all of \mathcal{C} at once. We also add a third wiring predicate, which has no analog within the GKR protocol: let $\text{io}: \{0, 1\}^{3k} \rightarrow \{0, 1\}$ denote the function that returns 1 when gate a is either a gate from the

explicit input x or one of the output gates, and gates b and c are the in-neighbors of a (input gates have in-neighbors $b = c = \mathbf{0}$).

Notice that add , mult , and io are independent of the inputs x and purported outputs y . The final function that plays a role in the MIP does depend on x and y . Define $I_{x,y}: \{0, 1\}^k \rightarrow \mathbb{F}$ such that $I_{x,y}(a) = x_a$ if a is the label of an input gate, $I_{x,y}(a) = y_a$ if a is the label of an output gate, and $I_{x,y}(a) = 0$ otherwise.

Lemma 7.3. *For $G_{x,y,W}(a, b, c): \{0, 1\}^{3k} \rightarrow \mathbb{F}$ defined as below, $G_{x,y,W}(a, b, c) = 0$ for all $(a, b, c) \in \{0, 1\}^{3k}$ if and only if W is a correct transcript for $\{C, x, y\}$:*

$$G_{x,y,W}(a, b, c) = \text{io}(a, b, c) \cdot (I_{x,y}(a) - W(a)) + \text{add}(a, b, c) \cdot (W(a) - (W(b) + W(c))) + \text{mult}(a, b, c) \cdot (W(a) - W(b) \cdot W(c)).$$

Proof. If W is not a correct transcript, there are five cases:

1. Suppose $a \in \{0, 1\}^k$ is the label of an input gate. If $W(a) \neq x_a$, then $G_{x,y,W}(a, 0, 0) = I_{x,y}(a) - W(a) = x_a - W(a) \neq 0$.
2. Suppose $a \in \{0, 1\}^k$ is the label of a non-output addition gate with in-neighbors b and c . If $W(a) \neq W(b) + W(c)$, then $G_{x,y,W}(a, b, c) = W(a) - (W(b) + W(c)) \neq 0$.
3. Suppose $a \in \{0, 1\}^k$ is the label of a non-output multiplication gate with in-neighbors b and c . If $W(a) \neq W(b) \cdot W(c)$, then $G_{x,y,W}(a, b, c) = W(a) - (W(b) \cdot W(c)) \neq 0$.
4. Suppose $a \in \{0, 1\}^k$ is the label of an output addition gate with in-neighbors b and c . If $y_a \neq W(b) + W(c)$, then $G_{x,y,W}(a, b, c) = I_{x,y}(a) - W(a) + (W(a) - (W(b) + W(c))) = y_a - (W(b) + W(c)) \neq 0$.
5. Suppose $a \in \{0, 1\}^k$ is the label of an output multiplication gate with in-neighbors b and c . If $y_a \neq W(b) \cdot W(c)$, then $G_{x,y,W}(a, b, c) = I_{x,y}(a) - W(a) + (W(a) - (W(b) \cdot W(c))) = y_a - (W(b) \cdot W(c)) \neq 0$.

On the other hand, if W is a correct transcript then it is immediate from the definition of $G_{x,y,W}$ that $G_{x,y,W}(a, b, c) = 0$ for all $(a, b, c) \in \{0, 1\}^{3k}$. \square

For any polynomial $Z: \mathbb{F}^k \rightarrow \mathbb{F}$, define the associated polynomial:

$$g_{x,y,Z}(a, b, c) = \tilde{\text{io}}(a, b, c) \cdot (\tilde{I}_{x,y}(a) - Z(a)) + \tilde{\text{add}}(a, b, c) \cdot (Z(a) - (Z(b) + Z(c))) + \tilde{\text{mult}}(a, b, c) \cdot (Z(a) - Z(b) \cdot Z(c)).$$

It follows from Lemma 7.3 that Z extends a correct transcript W if and only if $g_{x,y,Z}$ vanishes on the Boolean hypercube. We now define a polynomial $h_{x,y,Z}$ such that $g_{x,y,Z}$ vanishes on the Boolean hypercube if and only if $\sum_{u \in \{0, 1\}^{3k}} h_{x,y,Z}(u) = 0$.

Defining $h_{x,y,Z}$. As in Section 4.6.7.1, let $\beta_{3k}(a, b): \{0, 1\}^{3k} \times \{0, 1\}^{3k} \rightarrow \{0, 1\}$ be the function that evaluates to 1 if $a = b$, and evaluates to 0 otherwise, and define the formal polynomial

$$\tilde{\beta}_{3k}(a, b) = \prod_{j=1}^{3k} ((1 - a_j)(1 - b_j) + a_j b_j).$$

It is straightforward to check that $\tilde{\beta}_{3k}$ is the multilinear extension β_{3k} . Indeed, $\tilde{\beta}_{3k}$ is a multilinear polynomial. And for $a, b \in \{0, 1\}^{3k}$, it is easy to check that $\tilde{\beta}_{3k}(a, b) = 1$ if and only if a and b are equal coordinate-wise.

Consider the polynomial

$$p(X) := \sum_{Y \in \{0, 1\}^{3k}} \tilde{\beta}_{3k}(X, Y) \cdot g_{x,y,Z}(Y).$$

Clearly p is multilinear since $\tilde{\beta}$ is, and p vanishes on all inputs in $\{0, 1\}^{3k}$ if and only if $g_{x,y,Z}$ does. Since the multilinear extension on domain $\{0, 1\}^{3k}$ is unique, this means that p is the *identically zero* polynomial if and only if $g_{x,y,Z}$ vanishes on all inputs in $\{0, 1\}^{3k}$. For the verifier to check that p is indeed the zero-polynomial, it is enough for the verifier to pick a random input $r \in \{0, 1\}^{3k}$ and confirm that $p(r) = 0$ (because if p is any *nonzero* polynomial of total degree at most d , the Schwartz-Zippel lemma implies that $p(r)$ will equal 0 with probability at most $d/|\mathbb{F}|$).

Hence, we define

$$h_{x,y,Z}(Y) := \tilde{\beta}_{3k}(r, Y) \cdot g_{x,y,Z}(Y). \quad (7.2)$$

This definition ensures that $p(r) = \sum_{Y \in \{0,1\}^{3k}} h_{x,y,Z}(Y)$.

In summary, in the MIP, \mathcal{V} chooses r uniformly at random from the set \mathbb{F}^{3k} , defines $h_{x,y,Z}$ based on r as per Equation (7.2), and is convinced that Z extends a correct transcript for $\{\mathcal{C}, x, y\}$ as long as

$$0 = \sum_{Y \in \{0,1\}^{3k}} h_{x,y,Z}(r).$$

More formally, if $g_{x,y,Z}$ has total degree at most d , then with probability at least $1 - (d+1)/|\mathbb{F}|$ over the random choice of r , if $g_{x,y,Z}$ does not vanish on the Boolean hypercube then $\sum_{u \in \{0,1\}^{3k}} h_{x,y,Z}(u) \neq 0$. For simplicity, the remainder of the presentation ignores the $(d+1)/|\mathbb{F}|$ probability of error in this step (the $(d+1)/|\mathbb{F}|$ can be folded into the soundness error of the entire MIP).

Applying the Sum-Check Protocol to $h_{x,y,Z}$. \mathcal{V} applies the sum-check protocol to $h_{x,y,Z}$, with \mathcal{P}_1 playing the role of the prover in this protocol. To perform the final check in this protocol, \mathcal{V} needs to evaluate $h_{x,y,Z}$ at a random point $r \in \mathbb{F}^{3k}$. Let r_1, r_2, r_3 denote the first, second, and third k entries of r . Then evaluating $h_{x,y,Z}(r)$ requires evaluating $K_q(r)$, $\text{io}(r)$, $\widetilde{\text{add}}(r)$, $\widetilde{\text{mult}}(r)$, $\widetilde{\text{I}}_{x,y}(r_1)$, $Z(r_1)$, $Z(r_2)$, and $Z(r_3)$. \mathcal{V} can compute the first five evaluations without help in $O(\log(T))$ time, assuming that $\widetilde{\text{add}}$ and $\widetilde{\text{mult}}$ can be computed within this time bound (see Section 4.6.6 for further discussion of this assumption). However, \mathcal{V} cannot evaluate $Z(r_1)$, $Z(r_2)$, or $Z(r_3)$ without help. To deal with this, the verifier first uses the ‘‘Reducing the Verification of a Single Point’’ technique from Section 4.6 on the GKR protocol, to reduce the evaluation of Z at the three points r_1 , r_2 , and r_3 , to the evaluation of Z at a single point $r_4 \in \mathbb{F}^k$. To obtain the evaluation $Z(r_4)$, \mathcal{V} turns to \mathcal{P}_2 .

The Low-Degree Test. The paper [BTW14] uses the *point vs. plane* low-degree test, as analyzed by Moshkovitz and Raz [MR08]. It could have used the point-versus-line test from Chapter 6, and this would shave a logarithmic factor off of \mathcal{P}_2 ’s asymptotic time cost, as well as the total communication cost of the MIP (see Remark 7.3). However, the constants appearing in Arora and Sudan’s analysis of the point-versus-line test [AS03] are too large to yield a practical protocol, as the verifier would be forced to work over an enormous field to guarantee a suitable soundness error.

The point-versus-plane test involves two oracles Z, A' , with Z modeling the first prover in our MIP, and A' modeling the second. The oracle Z takes as input a $r_4 \in \mathbb{F}^k$ and returns some value $Z(r_4)$. Let \mathcal{Q} denote the set of all planes in \mathbb{F}^k . A' takes as input a plane $Q \in \mathcal{Q}$ returns some bivariate polynomial $A'(Q)$ of degree k in each variable (purported to be $Z \circ Q$ the restriction of Z to the plane specified by Q).

In the low-degree test, the verifier chooses a random point $r_4 \in \mathbb{F}^k$, and a random plane Q containing r_4 , and queries $A(r_4)$ and $A'(Q)$, accepting if and only if the claims of A and A' regarding $Z(r_4)$ are consistent.

Moshkovitz and Raz show that if the oracles pass the low-degree test with probability γ , then there is a polynomial Y of total degree at most k^2 such that A agrees with Y on a fraction of at least $\gamma - \varepsilon$ of the points

in \mathbb{F}^k , where $\varepsilon = 2^7 \cdot k (|\mathbb{F}|^{-1/8} + k^2 |\mathbb{F}^{-1/4}|)$. Note that $\varepsilon = o(1)$ if $|\mathbb{F}|$ is a sufficiently large polynomial in k , say $|\mathbb{F}| > k^{10}$.

In summary, if $|\mathbb{F}| > k^{10}$, and \mathcal{P}_1 and \mathcal{P}_2 pass the low-degree test with probability γ , then there is a polynomial Y of total degree at most k^2 that agrees with Z on a $\gamma - o(1)$ fraction of points in \mathbb{F}^k .

Remark 7.2. We do not cover the soundness analyses of low-degree tests such as point-vs-line or point-vs-plane in this manuscript, as these analyses are highly technical. As discussed earlier in this chapter, in the context of the MIP of this section, the low-degree test is effectively acting as a polynomial commitment scheme, and in practical argument systems, different techniques are used to design polynomial commitments.

The most closely related analyses covered in this survey can be found in Sections 8.4.3, 9.4 and 9.6, which are devoted to PCPs and IOPs for Reed-Solomon testing. A low-degree test such as point-vs-line or point-vs-plane confirms that a string is close to the evaluation table of a low-degree polynomial, without using any extra “help” from an untrusted prover during the confirmation procedure. A PCP or IOP for Reed-Solomon testing is essentially a low-degree test (for univariate polynomials) in which the prover helps the testing procedure. Another difference between low-degree tests and PCPs/IOPs for Reed-Solomon testing is that the latter exploit non-adaptivity assumptions on the prover that are not valid in the MIP setting—see Section 8.1 for details on the relationship between MIPs and PCPs/IOPs.

7.2.1.4 MIP Soundness Analysis

Theorem 7.4. *Suppose that \mathcal{P}_1 and \mathcal{P}_2 convince the MIP verifier to accept with probability $\gamma > .5 + \varepsilon$ for $\varepsilon = \Omega(1)$. Then there is some polynomial Y such that $h_{x,y,Y}$ satisfies Equation (7.1).*

Detailed Sketch. Since \mathcal{P}_1 and \mathcal{P}_2 pass the low-degree test with probability at least γ , the low-degree test guarantees that there is some polynomial Y of total degree at most k^2 such that Z and Y agree on a $p \geq \gamma - o(1)$ fraction of points. Here, Z is the function specified by the oracle modeling \mathcal{P}_1 in the low-degree test. Since Y has total degree at most k^2 , $h_{x,y,Y}$ has total degree at most $2k^2 + 6k$.

Suppose that $h_{x,y,Y}$ does not satisfy Equation (7.1). Let us say that \mathcal{P}_1 *cheats* at round i of the sum-check protocol if he does not send the message that is prescribed by the sum-check protocol in that round, when applied to the polynomial $h_{x,y,Y}$. The soundness analysis of the sum-check protocol (Section 4.1) implies that if \mathcal{P}_1 falsely claims that $h_{x,y,Y}$ does satisfy Equation (7.1), then with probability at least $1 - 3k \cdot (2k^2 + 6k) / |\mathbb{F}| = 1 - o(1)$, \mathcal{P}_1 will be forced to cheat at all rounds of the sum-check protocol including the last one. This means that in the last round, \mathcal{P}_1 sends a message that is inconsistent with the polynomial Y .

If \mathcal{P}_1 does cheat in the last round, the verifier will reject unless, in the final check of the protocol, the verifier winds up choosing a point in \mathbb{F}^{3k} at which $h_{x,y,Y}$ and $h_{x,y,Z}$ disagree. This only happens if \mathcal{V} picks a point $r \in \mathbb{F}^k$ for use in the low-degree test such that $Y(r) \neq Z(r)$. But this occurs with probability only $1 - p = 1 - \gamma + o(1)$. In total, the probability that \mathcal{P}_1 passes all tests within the sum-check protocol is therefore at most $1 - \gamma + o(1)$. If $\gamma > \frac{1}{2}$, this contradicts the fact that \mathcal{P}_1 and \mathcal{P}_2 convince the MIP verifier to accept with probability at least γ . \square

Recall that if $h_{x,y,Y}$ satisfies Equation (7.1), then $g_{x,y,Y}$ vanishes on the Boolean hypercube, and hence Y is an extension of a correct transcript for $\{\mathcal{C}, x, y\}$. So Theorem 7.4 implies that if the MIP verifier accepts with probability $\gamma > \frac{1}{2}$, then there is a correct transcript for $\{\mathcal{C}, x, y\}$.

Although the soundness error can be reduced from $\frac{1}{2} + o(1)$ to an arbitrarily small constant with $O(1)$ independent repetitions of the MIP, this would be highly expensive in practice. Fortunately, [BTVW14]

Communication	Rounds	\mathcal{V} time	\mathcal{P}_1 and \mathcal{P}_2 time
$O(\log S)$ field elements	$O(\log S)$	$O(n + \text{polylog}(S))$	$O(S \log S)$

Table 7.1: Costs of the MIP of Section 7.2, when using the point-vs-line low-degree test. The stated costs assume that $\widetilde{\text{add}}$, $\widetilde{\text{mult}}$, and $\widetilde{\text{io}}$ can be evaluated in time $\text{polylog}(S)$. As explained in Section 5.5, this holds for the circuits generated by reductions from RAM simulation.

performs a more careful soundness analysis that establishes that the MIP itself, *without repetition*, has soundness error $o(1)$.

The bottleneck in the soundness analysis of Theorem 7.4 that prevents the establishment of soundness error less than $\frac{1}{2}$ is that, if the prover’s pass the low-degree test with probability $\gamma < \frac{1}{2}$, then one can only guarantee that there is a polynomial Y that agrees with Z on a γ fraction of points. The verifier will choose a random point r in the sum-check protocol at which Y and Z disagree with probability $1 - \gamma > \frac{1}{2}$, and in this case all bets are off.

The key to the stronger analysis is to use a stronger guarantee from the low-degree test, known as a *list-decoding guarantee*. Roughly speaking, the list-decoding guarantee ensures that if the oracles pass the low-degree test with probability γ , then there is a “small” number of low-degree polynomials Q_1, Q_2, \dots that “explain” essentially all of the tester’s acceptance, in the sense that for almost all points r at which the low-degree test passes, $A(r)$ agrees with $Q_i(r)$ for at least one i . This allows one to argue that even if the provers pass the low-degree test with probability only $\gamma < \frac{1}{2}$, the sum-check protocol will still catch \mathcal{P}_1 in a lie with probability very close to 1.

7.2.1.5 Protocol Costs

Verifier’s Costs. \mathcal{V} and \mathcal{P}_1 exchanges two messages for each variable of $h_{x,y,z}$, and where \mathcal{P}_2 exchanges two messages in total with \mathcal{V} . This is $O(\log S)$ messages in total. Each message from \mathcal{P}_1 is a polynomial of degree $O(1)$, while the message from \mathcal{P}_2 is a bivariate polynomial of total degree $O(\log S)$. In total, all messages can be specified using $O(\log^2 S)$ field elements (the bottleneck is \mathcal{P} ’s message). As for \mathcal{V} ’s runtime, the verifier has to process the provers’ messages, and then to perform the last check in the sum-check protocol, she must evaluate $\widetilde{\text{add}}$, $\widetilde{\text{mult}}$, $\widetilde{\text{io}}$, and $\widetilde{\text{I}}$ at random points. The verifier requires $O(\log^2 S)$ time to process the provers’ messages, and Lemma 3.8 implies that \mathcal{V} can evaluate $\widetilde{\text{I}}$ at a random point in $O(n)$ time. We assume that $\widetilde{\text{add}}$, $\widetilde{\text{mult}}$, and $\widetilde{\text{io}}$ can be evaluated at a point in time $\text{polylog}(S)$ as well— as explained in Section 5.5, this (essentially) holds for the circuits generated by reductions from RAM simulation.

Prover’s Costs. Blumberg et al. [BTVW14] showed that, using the techniques developed to implement the prover in the GKR protocol (Section 4.6), specifically Method 2 described there, \mathcal{P}_1 can be implemented in $O(S \log S)$ time. In fact, using more advanced techniques (e.g., Lemma 4.5), it is possible to implement the first prover in $O(S)$ time (the interested reader is referred to [Set19] for details). \mathcal{P}_2 needs to specify $\widetilde{W} \circ Q$, where Q is a random plane in \mathbb{F}^k . It suffices for \mathcal{P}_2 to evaluate \widetilde{W} at $O(\log^2 S)$ many points—using Lemma 3.8, this can be done in $O(S)$ time per point, resulting in a total runtime of $O(S \log^2 S)$.

Remark 7.3. If the point-vs-line test were used in place of the point vs. plane test, \mathcal{P}_2 would only need to evaluate \widetilde{W} at $O(\log S)$ points, which can be done in $O(S \log S)$ time, and the total communication cost would drop from $O(\log^2 S)$ to $O(\log S)$.

7.3 A Succinct Argument for Deep Circuits

Using any polynomial commitment scheme, one can turn the MIP of the previous section into a succinct argument for deep and narrow arithmetic circuits.⁸⁵ Specifically, one gets rid of the second prover, and instead just had the first prover commit to \tilde{W} at the start of the protocol. At the end of the verifier’s interaction with the first prover in the MIP above, the first prover makes a claim about $\tilde{W}(r_4)$, which the verifier checks directly by having the prover reveal it via the polynomial commitment protocol.

This succinct argument has an advantage over the approach to succinct argument from Chapter 6 that was based directly on the GKR protocol: namely, the argument system based on the MIP of the previous section is succinct with a nearly-linear time verifier *even for deep and narrow circuits*. The *disadvantage* of the argument system from the previous section is that it applies the polynomial commitment scheme to the entire *transcript extension* $\tilde{W}: \mathbb{F}^{\log|\mathcal{C}|} \rightarrow \mathbb{F}$, whereas the argument system of Chapter 6 applied the polynomial commitment scheme only to the multilinear extension of the *witness* \tilde{w} . The expense of applying a commitment scheme to \tilde{w} will be much smaller than the expense of applying it to \tilde{W} if the the witness size $|w|$ is much smaller than the circuit size $|\mathcal{C}|$.

Existing polynomial commitment schemes are still the concrete bottlenecks in argument systems that use them [Set19]. Since the witness w can be much smaller than circuit \mathcal{C} , applying the polynomial commitment scheme to \tilde{w} can be significantly less expensive than applying it to \tilde{W} (so long as the witness makes up only a small fraction of the total number of gates in the circuit). Besides, we’ve seen that short, wide circuits are “universal” in the context of succinct arguments, since any RAM running in time T can be turned into an instance of arithmetic circuit satisfiability of size close to T and depth close to $O(\log T)$. In summary, which approach yields a superior argument system for circuit satisfiability in practice depends on many factors, including witness size, circuit depth, the relative importance of proof length vs. other protocol costs, etc.

Remark 7.4. Bitansky and Chiesa [BC12] gave a different way to transform MIPs into succinct arguments, but their transformation used (multiple layers of) fully homomorphic encryption, rendering it highly impractical. Unlike the MIP-to-argument transformation in this section, Bitansky and Chiesa’s transformation works for *arbitrary* MIPs. The transformation in this section exploits additional structure of the specific MIP of this section, specifically the fact that the sole purpose of the second prover in the MIP is to run a low-degree test. In the setting of succinct arguments, this role played by the second prover can be replaced with a polynomial commitment scheme. In summary, while Bitansky and Chiesa’s transformation from MIPs to arguments is more general (applying to arbitrary MIPs, not just those in which the second prover is solely used to run a low-degree test), it is much less efficient than the transformation of this section.

7.4 Preview: A General Paradigm for the Design of Succinct Arguments

The succinct argument described in Section 7.3 has the following two-stage form. First, the prover cryptographically commits to a multilinear polynomial p that is claimed to extend a valid transcript for the claim that there is a witness w such that $\mathcal{C}(x, w) = y$. Second, the verifier probabilistically checks that the polynomial p indeed extends a correct transcript.

This paradigm is repeated over and over again in the remainder of this manuscript. The main difference between the various protocols is the *type* of low-degree extension polynomial used. In the argument system of Section 7.3, p is a multilinear polynomial over roughly $\log|\mathcal{C}|$ variables. In the PCP-based and IOP-based

⁸⁵The polynomial commitment scheme should be extractable in addition to binding. See Section 6.3.3 for details.

arguments of Sections 8.4 and 9, p is a *univariate* polynomial of degree roughly $|\mathcal{C}|$. In the linear-PCP-based arguments of Section 14, p is a *linear* polynomial over $|\mathcal{C}|$ variables, meaning p outputs some linear combination of its inputs (equivalently, p has total degree 1).

Based on which type of polynomial is used, the polynomial commitment scheme may have to change, as do the techniques the verifier uses to probabilistically checking that p extends a valid transcript for the claim that $\mathcal{C}(x, w) = y$. Even though details differ depending on which type of polynomial is used, many ideas recur between settings. As we will see, some polynomial commitment schemes actually solve a more general problem, allowing the evaluation of arbitrary inner products over committed vectors; both univariate and multivariate polynomial commitments are special cases of this problem. Occasionally, it can even be the case that one “combines” the use of two different types of polynomials, by, e.g., using a polynomial commitment scheme for univariate polynomials as a subroutine in a polynomial commitment scheme for multilinear polynomials (see, e.g., Section 9.5).

7.5 Extension from Circuit-SAT to R1CS-SAT

Chapter 5 gave techniques for turning computer programs into equivalent instances of arithmetic circuit satisfiability, and Chapter 6 and this chapter gave succinct non-interactive arguments for arithmetic circuit satisfiability. Arithmetic circuit satisfiability is an example of an *intermediate representation*, a term that refers to any model of computation that is directly amenable to application of interactive proof or argument systems.

A related intermediate representation that has proven popular and convenient in practice is *rank-1 constraint system* (R1CS) instances (also often referred to as *Quadratic Arithmetic Programs* (QAPs)). An R1CS instance is specified by three $m \times n$ matrices A, B, C with entries from a field \mathbb{F} and is satisfiable if and only if there is a vector $z \in \mathbb{F}^n$ such that

$$(A \cdot z) \circ (B \cdot z) = C \cdot z. \tag{7.3}$$

Here, \cdot denotes matrix-vector product, and \circ denotes entrywise (a.k.a. Hadamard) product. Any vector z satisfying Equation (7.3) is analogous to the notion of a “correct transcript” in the context of arithmetic circuit satisfiability (Section 7.2.1.1).

7.5.1 Relationship Between R1CS-SAT and Arithmetic Circuit-SAT

The R1CS-SAT problem can be thought of as a generalization of the Arithmetic Circuit-SAT problem in the following sense: Any instance of Arithmetic Circuit-SAT can be efficiently transformed into instances of R1CS-SAT. The number of rows and columns of the matrices appearing in the resulting R1CS instance is proportional to the number of gates in \mathcal{C} , and the number of nonzero entries in any row of the matrices is bounded above by the fan-in of the circuit \mathcal{C} . For fan-in two circuits, this means that the equivalent R1CS-SAT instances are sparse, and hence we will ultimately seek protocols where the prover(s) run in time proportional to the number of nonzero entries of these matrices.

To see this, consider an instance $\{\mathcal{C}, x, y\}$ of arithmetic circuit-SAT, i.e., where the prover wants to convince the verifier that there is a w such that $\mathcal{C}(x, w) = y$. We need to construct matrices A, B, C such that there exists a vector z such that Equation (7.3) holds if and only if the preceding sentence is true.

Let N be the sum of the lengths of x, y , and w , plus the number of gates in \mathcal{C} , and let $M = N - |w|$. The R1CS-SAT instance will consist of three $M \times (N + 1)$ matrices A, B , and C . We will fix the first entry z_1 of z to 1, and associate each remaining entry of z with either an entry of x, y , or w , or a gate of \mathcal{C} .

For an entry j of z corresponding to an entry x_i of x , we define the j th row of A, B, C to capture the constraint the z_j must equal x_i . That is, we set the j th row of A to be the standard basis vector $e_1 \in \mathbb{F}^{N+1}$, the j th row of B to be the standard basis vector $e_j \in \mathbb{F}^{N+1}$ and the j th row of C to be $x_i \cdot e_1$. This means that the j th constraint in the RICS system asserts that $z_j - x_i = 0$ which is equivalent to demanding that $z_j = x_i$. We include an analogous constraint for each entry j of z corresponding to an entry of y .

For each entry j of z corresponding to an addition gate of \mathcal{C} (with in-neighbors indexed by $j', j'' \in \{2, \dots, N+1\}$), we define the j th row of A, B, C to capture the constraint the z_j must equal the sum of the two inputs to that addition gate. That is, we set the j th row of A to be the standard basis vector $e_1 \in \mathbb{F}^{N+1}$, the j th row of B to be $e_{j'} + e_{j''} \in \mathbb{F}^{N+1}$ and the j th row of C to be $e_{j'}$. This means that the j th constraint in the RICS system asserts that $(z_{j'} + z_{j''}) - z_j = 0$ which is equivalent to demanding that $z_j = z_{j'} + z_{j''}$.

Finally, for each entry j of z corresponding to a multiplication gate of \mathcal{C} (with in-neighbors indexed by $j', j'' \in \{2, \dots, N+1\}$), we define the j th row of A, B, C to capture the constraint the z_j must equal the product of the two inputs to that addition gate. That is, we set the j th row of A to be the standard basis vector $e_{j'} \in \mathbb{F}^{N+1}$, the j th row of B to be the standard basis vector $e_{j''} \in \mathbb{F}^{N+1}$ and the j th row of C to be $e_{j'} + e_{j''}$. This means that the j th constraint in the RICS system asserts that $(z_{j'} \cdot z_{j''}) - z_j = 0$ which is equivalent to demanding that $z_j = z_{j'} \cdot z_{j''}$.

7.5.2 A MIP for RICS-SAT

As observed in [Set19], we can apply the ideas of this chapter to give a MIP and associated succinct argument for RICS instances. View the matrices A, B, C as functions $f_A, f_B, f_C: \{0, 1\}^{\log_2 m} \times \{0, 1\}^{\log_2 n} \rightarrow \mathbb{F}$ in the natural way as per Sections 4.3 and 4.4. Just as in the MIP of this chapter (Section 7.2.1.2), the prover claims to hold an extension polynomial Z of a correct transcript z for the RICS instance. Observe that a polynomial $Z: \mathbb{F}^{\log_2 m} \times \mathbb{F}^{\log_2 n} \rightarrow \mathbb{F}$ extends a correct transcript z for the RICS instance if and only if the following equation holds for all $a \in \{0, 1\}^{\log_2 m}$:

$$\left(\sum_{b \in \{0, 1\}^{\log_2 n}} \tilde{f}_A(a, b) \cdot Z(b) \right) \cdot \left(\sum_{b \in \{0, 1\}^{\log_2 n}} \tilde{f}_B(a, b) \cdot Z(b) \right) - \left(\sum_{b \in \{0, 1\}^{\log_2 n}} \tilde{f}_C(a, b) \cdot Z(b) \right) = 0. \quad (7.4)$$

Let g_Z denote the $(\log_2(m))$ -variate polynomial

$$g_Z(X) = \left(\sum_{b \in \{0, 1\}^{\log_2 n}} \tilde{f}_A(X, b) \cdot Z(b) \right) \cdot \left(\sum_{b \in \{0, 1\}^{\log_2 n}} \tilde{f}_B(X, b) \cdot Z(b) \right) - \left(\sum_{b \in \{0, 1\}^{\log_2 n}} \tilde{f}_C(X, b) \cdot Z(b) \right) \quad (7.5)$$

This polynomial has degree at most 2 in each variable (i.e., it is multi-quadratic), and Equation (7.4) holds if and only if g_Z vanishes at all inputs in $\{0, 1\}^{\log_2 m}$.

We obtain a MIP for checking that g_Z vanishes over the Boolean hypercube in a manner analogous to Section 7.2.1.3. Specifically, we can define a related polynomial h_Z by picking a random point $r \in \{0, 1\}^{\log_2 m}$ and, in analogy with Equation (7.2), defining

$$h_Z(Y) = \tilde{\beta}_{\log_2 m}(r, Y) \cdot g_Z(Y).$$

Following the reasoning preceding Equation (7.2), by the Schwartz-Zippel Lemma, it holds that, up to a negligible soundness error (at most $\log_2(m)/|\mathbb{F}|$), g_Z vanishes on the Boolean hypercube if and only if

$$\sum_{a \in \{0, 1\}^{\log_2 m}} h_Z(a) = 0.$$

The verifier can compute this last expression by applying the sum-check protocol to the polynomial

$$h_Z(Y) = \tilde{\beta}_{\log_2 m}(r, Y) \cdot g_Z(Y).$$

After applying the sum-check protocol to $h_Z(Y)$, the verifier needs to evaluate $h_Z(Y)$ at a random input $r' \in \mathbb{F}^{\log_2 m}$. To evaluate $h_Z(r')$, it is enough for the verifier to evaluate $\tilde{\beta}_{\log_2 m}(r, r')$ and $g_Z(r')$. The former quantity can be evaluated by the verifier in $O(\log_2 m)$ operations in \mathbb{F} using Equation (4.17). The verifier cannot efficiently evaluate $g_Z(r')$ on its own, but by definition (Equation (7.5)), this quantity equals:

$$\left(\sum_{b \in \{0,1\}^{\log_2 n}} \tilde{f}_A(r', b) \cdot Z(b) \right) \cdot \left(\sum_{b \in \{0,1\}^{\log_2 n}} \tilde{f}_B(r', b) \cdot Z(b) \right) - \left(\sum_{b \in \{0,1\}^{\log_2 n}} \tilde{f}_C(r', b) \cdot Z(b) \right). \quad (7.6)$$

This means that to compute $g_Z(r')$, it suffices to apply the sum-check protocol three more times, to the following three $(\log_2(n))$ -variate polynomials:

$$\begin{aligned} p_1(X) &= \tilde{f}_A(r', X) \cdot Z(X). \\ p_2(X) &= \tilde{f}_B(r', X) \cdot Z(X). \\ p_3(X) &= \tilde{f}_C(r', X) \cdot Z(X). \end{aligned}$$

This is because applying the sum-check protocol to $p_1(X)$ computes

$$\left(\sum_{b \in \{0,1\}^{\log_2 n}} \tilde{f}_A(r', b) \cdot Z(b) \right)$$

and similarly applying the sum-check protocol to p_2 and p_3 computes the remaining two quantities appearing in Equation (7.6). As a concrete optimization, all three invocations of sum-check can be executed in parallel, using the same randomness in each of the three invocations.

At the end of these three final invocations of the sum-check protocol, the verifier needs to evaluate each of p_1, p_2, p_3 at a random input r'' . To accomplish this, it suffices for the verifier to evaluate $\tilde{f}_A(r', r'')$, $\tilde{f}_B(r', r'')$, $\tilde{f}_C(r', r'')$, and $Z(r'')$.

At this point, the situation is exactly analogous to the MIP for arithmetic circuit-SAT of Section 7.2.1.3, with \tilde{f}_A , \tilde{f}_B , and \tilde{f}_C playing the roles of the “wiring predicates” $\widetilde{\text{add}}$ and $\widetilde{\text{mult}}$. That is, for many natural RICS systems, the verifier can evaluate \tilde{f}_A , \tilde{f}_B , and \tilde{f}_C in logarithmic time unaided, and $Z(r'')$ can be obtained from the second prover using a low-degree test

Regarding the prover’s runtime, we claim that the first prover in the MIP, if given a satisfying assignment $z \in \mathbb{F}^n$ for the RICS instance, can be implemented in time proportional to the number K of nonzero entries of the matrices A , B , and C (here, we assume without loss of generality that this number is at least $n + m$, i.e., no row or column of any matrix is all zeros).

We begin by showing that in the first invocation of the sum-check protocol within the MIP, to the polynomial $h_Z(Y)$, the prover can be implemented in time proportional to the number of nonzero entries of A , B , and C . This holds by the following reasoning. First, observe that

$$h_Z(Y) = \tilde{\beta}_{\log_2 m}(r, Y) \cdot g_Z(Y) = \tilde{\beta}_{\log_2 m}(r, Y) \cdot q_1(Y) \cdot q_2(Y) - \tilde{\beta}_{\log_2 m}(r, Y) \cdot q_3(Y), \quad (7.7)$$

where

$$\begin{aligned}
q_1(Y) &= \left(\sum_{b \in \{0,1\}^{\log_2 n}} \tilde{f}_A(Y,b) \cdot Z(b) \right), \\
q_2(Y) &= \left(\sum_{b \in \{0,1\}^{\log_2 n}} \tilde{f}_B(Y,b) \cdot Z(b) \right), \\
q_3(Y) &= \left(\sum_{b \in \{0,1\}^{\log_2 n}} \tilde{f}_C(Y,b) \cdot Z(b) \right).
\end{aligned}$$

We wish to apply Lemma 4.5 to conclude that the prover in the sum-check protocol applied the h_Z can be implemented quickly. Since $\tilde{\beta}_{\log_2 m}(r, Y)$, $q_1(Y)$, $q_2(Y)$, and $q_3(Y)$ are all multilinear polynomials in the variables Y , to apply Lemma 4.5, it is enough to show that all four of these multilinear polynomials can be evaluated at all inputs $a \in \{0, 1\}^{\log_2 m}$ in time proportional to the number of nonzero entries of A , B , and C .⁸⁶

First, we observe that $\tilde{\beta}_{\log_2 m}(r, a)$ can be evaluated by the prover at all inputs $a \in \{0, 1\}^{\log_2 m}$ in $O(m)$ total time, as this task is equivalent to evaluating all $(\log m)$ -variate Lagrange basis polynomials at input $r \in \mathbb{F}^{\log m}$, which the proof of Lemma 3.8 revealed is possible to achieve in $O(m)$ time.

Second, we turn to the claim that q_1 , q_2 , and q_3 can be evaluated at all inputs $a \in \{0, 1\}^{\log_2 m}$ in the requisite time bound. This holds because, if we interpret $a \in \{0, 1\}^{\log_2 m}$ as a number in $\{1, \dots, m\}$ and let A_a , B_a , and C_a respectively denote the a th row of A , B , and C , then $q_1(a)$ is simply $A_a \cdot z$, and similarly $q_2(a) = B_a \cdot z$ and $q_3(a) = C_a \cdot z$. Hence all three polynomials can be evaluated at all $a \in \{0, 1\}^{\log_2(m)}$ in time proportional to the number of nonzero entries of the three matrices A , B , and C .

Similar observations reveal that the prover in the three invocations of the sum-check protocol applied to p_1 , p_2 , and p_3 can also be implemented in time proportional to the number of nonzero entries of A , B , and C . For example, $p_1(X)$ is a product of two multilinear polynomials $\tilde{f}_A(r', X)$ and $Z(X)$. To apply Lemma 4.5, the evaluations of $Z(X)$ at all inputs $b \in \{0, 1\}^{\log_2 n}$ are directly given by the satisfying assignment vector z for the RICS instance. Turning to $\tilde{f}_A(r', X)$, let $v \in \mathbb{F}^n$ denote the vector of all $(\log_2 n)$ -variate Lagrange basis polynomials evaluated at r' . Note that the proof of Lemma 3.8 shows that the vector v can be computed in $O(n)$ time. It can be seen that for $b \in \{0, 1\}^{\log_2 n}$, $\tilde{f}_A(r', b)$ is just the inner product of v with the b 'th column of A , which (given v) can be computed in time proportional to the number of nonzero entries in this column of A . This completes the explanation of why $p_1(X)$ can be evaluated at all inputs $b \in \{0, 1\}^{\log_2 n}$ in time proportional to the number of nonzero entries of A , and similarly for $p_2(X)$ and $p_3(X)$ (with A replaced with B and C respectively).

7.6 MIP = NEXP

In the MIP for arithmetic circuit-SAT of Section 7.2, if the circuit has size S then the verifier's runtime is $\text{poly}(\log S)$, plus the time required to evaluate $\widetilde{\text{add}}$, $\widetilde{\text{mult}}$, and $\widetilde{\text{io}}$ at random inputs. The transformation from computer programs to circuit-SAT instances sketched in Section 5.5 transforms any non-deterministic random access machine running in time T into an arithmetic circuit of size $\tilde{O}(T)$ in which $\widetilde{\text{add}}$, $\widetilde{\text{mult}}$, and $\widetilde{\text{io}}$

⁸⁶Equation (7.7) represents h_Z as a sum two products of $O(1)$ multilinear polynomials, while Lemma 4.5 as stated applies only to products of $O(1)$ multilinear polynomials directly. But the lemma extends easily to sums of polynomials, because the honest prover's messages in the sum-check protocol applied to a sum of two polynomials p and q is just the sum of the messages when the sum-check protocol is applied to p and q individually.

can be evaluated at any desired point in time $O(\log T)$. This means that the verifier in the MIP applied to the resulting circuit runs in polynomial time as long as $T \leq 2^{n^c}$ for some constant $c > 0$. In other words, the class of problems solvable in non-deterministic exponential time (**NEXP**) is contained in **MIP**, the class of languages solvable by a multi-prover interactive proof with a polynomial time verifier [BFL91].

The other inclusion, that **MIP** \subseteq **NEXP**, follows from the following simple reasoning. Given any multi-prover interactive proof system for a language \mathcal{L} and input x , one can in non-deterministic exponential time calculate the acceptance probability of the optimal strategy available to provers attempting to convince the verifier to accept, as follows. First, non-deterministically guess the optimal strategies of the provers. Second, compute the acceptance probability that the strategy induces by enumerating over all possible coin tosses of the verifier and seeing how many lead to acceptance when interacting with the optimal prover strategy [FRS88]. Since the multi-prover interactive proof system is a valid MIP for \mathcal{L} this acceptance probability is at least $2/3$ if and only if $x \in \mathcal{L}$.

Chapter 8

PCPs and Succinct Arguments

8.1 PCPs: Definitions and Relationship to MIPs

In a MIP, if a prover is asked multiple questions by the verifier, then the prover can behave adaptively, which means that the prover's responses to any question can depend on the earlier questions asked by the verifier. This adaptivity was potentially bad for soundness, because the prover's ability to behave adaptively makes it harder to “pin down” the prover(s) in a lie. But, as will become clear below, it was potentially good for efficiency, since an adaptive prover can be asked a sequence of questions, and only needs to “think up” answers to questions that are actually asked.

In contrast, Probabilistically Checkable Proofs (PCPs) have non-adaptivity baked directly into the definition, by considering a verifier \mathcal{V} who is given oracle access to a static proof string π . Since π is static, \mathcal{V} can ask several queries to π , and π 's response to any query q_i can depend only on q_i , and not on q_j for $j \neq i$.

Definition 8.1. A probabilistically checkable proof system (PCP) for a language $\mathcal{L} \subseteq \{0, 1\}^*$ consists of a probabilistic polynomial time verifier \mathcal{V} who is given access to an input x , and oracle access to a proof string $\pi \in \Sigma^\ell$. The PCP has completeness error δ_c and soundness error δ_s if the following two properties hold.

1. (*Completeness*) For every $x \in \mathcal{L}$, there exists a proof string $\pi \in \Sigma^\ell$ such that $\Pr[\mathcal{V}^\pi(x) = \text{accept}] \geq 1 - \delta_c$.
2. (*Soundness*) For every $x \notin \mathcal{L}$ and every proof string $\pi \in \Sigma^\ell$, $\Pr[\mathcal{V}^\pi(x) = \text{accept}] \leq \delta_s$.

ℓ is referred to as the *length* of the proof, and Σ as the alphabet used for the proof. We think of all of these parameters as functions of the input size n . We refer to the time required to *generate* the honest proof string π as the prover time of the PCP.

Remark 8.1. The PCP model was introduced by Fortnow, Rompel, and Sipser [FRS88], who referred to it as the “oracle” model (we used this terminology in Lemma 7.2). The term Probabilistically Checkable Proofs was coined by Arora and Safra [AS98].

Remark 8.2. Traditionally, the notation $\mathbf{PCP}_{\delta_c, \delta_s}[r, q]_\Sigma$ is used to denote the class of languages that have a PCP verifier with completeness error δ_c , soundness error δ_s , and in which the verifier uses at most r random bits, and makes at most q queries to a proof string π over alphabet Σ . This notation is motivated in part by the importance of the parameters r , q , and Σ in applications to hardness of approximation. In the setting of verifiable computing, the most important costs are typically the verifier's and prover's runtime, and the total number q of queries (since, when PCPs are transformed into succinct arguments, the proof length of

the argument is largely determined by q). Note however that the proof length ℓ is a *lower bound* on the prover's runtime in any PCP system (since it takes time at least ℓ to write down a proof of length ℓ). Hence, obtaining a PCP with a small proof length is necessary, but not sufficient, for developing a PCP system with an efficient prover.

PCPs and MIPs are closely related: any MIP can be turned into a PCP, and vice versa. However, both transformations can result in a substantial increase in costs. The easier direction is turning a MIP into a PCP. This simple transformation dates back to Fortnow, Rompel, and Sipser, who introduced the PCP model, albeit under a different name.

Lemma 8.2. *Suppose $\mathcal{L} \subseteq \{0, 1\}^*$ has a k -prover MIP in which \mathcal{V} sends exactly one message to each prover, with each message consisting of at most r_Q bits, and each prover sends at most r_A bits in response to the verifier. Then \mathcal{L} has a k -query PCP system over an alphabet Σ of size 2^{r_A} , where the proof length is $k \cdot 2^{r_Q}$, with the same verifier runtime and soundness and completeness errors as the MIP.*

Sketch. For every prover \mathcal{P}_i in the MIP, the PCP proof has an entry for every possible message that \mathcal{V} might send to \mathcal{P}_i . The entry is equal to the prover's response to that message from \mathcal{V} . The PCP verifier simulates the MIP verifier, treating the proof entries as prover answers in the MIP. \square

Remark 8.3. It is also straightforward to obtain a PCP from a k -prover MIP in which \mathcal{V} sends multiple messages to each prover. If each prover \mathcal{P}_i is sent z messages $m_{i,1}, \dots, m_{i,z}$ in the MIP, obtain a new MIP by replacing \mathcal{P}_i with z provers $\mathcal{P}_{i,1}, \dots, \mathcal{P}_{i,z}$ who are each sent one message (the message to $\mathcal{P}_{i,j}$ being the concatenation of $m_{i,1}, \dots, m_{i,j}$).⁸⁷ The verifier in the $(z \cdot k)$ -prover MIP simulates the verifier in the k -prover MIP, treating $\mathcal{P}_{i,j}$'s answer as if it were \mathcal{P}_i 's answer to $m_{i,j}$. Completeness of the resulting $(z \cdot k)$ -prover MIP follows from completeness of the original k -prover MIP, by having prover $\mathcal{P}_{i,j}$ answer the same as \mathcal{P}_i would upon receiving message $m_{i,j}$. Soundness of the resulting $(z \cdot k)$ -prover MIP is implied by soundness of the original k -prover MIP.

Finally, apply Lemma 8.2 to the resulting $(z \cdot k)$ -prover MIP.

Lemma 8.2 highlights a fundamental difference between MIPs and PCPs: in a PCP, the prover must pre-compute a response for every possible query of the verifier, which will result in a very large prover runtime unless the number of possible queries from the verifier is small. Whereas in a MIP, the provers only need to compute responses “on demand”, ignoring any queries that the verifier *might* have asked, but did not. Hence, the MIP \implies PCP transformation of Lemma 8.2 may cause a huge blowup in prover runtime.

Lemma 7.2 gave a transformation from a PCP to a 2-prover MIP, but this transformation was also expensive. In summary, the tasks of constructing efficient MIPs and PCPs are incomparable. On the one hand, PCP provers are inherently non-adaptive, but they must pre-compute the answers to all possible queries of the verifier. MIP provers only need to compute answers “on demand”, but they can behave adaptively, and while there are generic techniques to force them to behave non-adaptively, these techniques are expensive.

8.2 Compiling a PCP Into a Succinct Argument

We saw in Chapter 6 that one can turn the GKR interactive proof for arithmetic circuit *evaluation* into a succinct argument for arithmetic circuit *satisfiability* (recall that the goal of a circuit satisfiability instance

⁸⁷The reason $\mathcal{P}_{i,j}$ must be sent the concatenation of the first j messages to \mathcal{P}_i rather than just $m_{i,j}$ is to ensure completeness of the resulting $(z \cdot k)$ -prover MIP. \mathcal{P}_i 's answer to $m_{i,j}$ is allowed to depend on all preceding messages $m_{i,1}, \dots, m_{i,j-1}$. So in order for $\mathcal{P}_{i,j}$ to be able to determine \mathcal{P}_i 's answer to $m_{i,j}$, it may be necessary for $\mathcal{P}_{i,j}$ to know $m_{i,1}, \dots, m_{i,j-1}$.

$\{C, x, y\}$ is to determine whether there exists a witness w such that $C(x, w) = y$). At the start of the argument, the prover sends a cryptographic commitment to the multilinear extension \tilde{w} of a witness w . The prover and verifier then run the GKR protocol to check that $C(x, w) = y$. At the end of the GKR protocol, the prover is forced to make a claim about the value of $\tilde{w}(r)$ for a random point r . The argument system verifier confirms that this claim is consistent with the corresponding claim derived from the cryptographic commitment to \tilde{w} .

The polynomial commitment scheme described in Section 6.3.2.2 consisted of two pieces; a *string-commitment* scheme using a Merkle tree, which allowed the prover to commit to *some* fixed function claim to equal \tilde{w} , and a low-degree test, which allowed the verifier to check that the function committed to was indeed (close to) a low-degree polynomial.

If our goal is to transform a PCP rather than an interactive proof into a succinct argument, we can use a similar approach, but omit the low-degree test. Specifically, as explained below, Kilian [Kil92] famously showed that any PCP can be combined with Merkle-hashing to yield four-message argument systems for all of **NP**, assuming that collision-resistant hash functions exist. The prover and verifier runtimes are the same as in the underlying PCP, up to low-order factors, and the total communication cost is $O(\log n)$ cryptographic hash values per PCP query. Micali [Mic00] showed that applying the Fiat-Shamir transformation to the resulting four-message argument system yields a *non-interactive* succinct argument in the random oracle model.⁸⁸

The idea is the following. The argument system consists of two phases: commit and reveal. In the commit phase, the prover writes down the PCP π , but doesn't send it to the verifier. Instead, the prover builds a Merkle tree, with the symbols of the PCP as the leaves, and sends the root hash of the tree to the verifier. This binds the prover to the string π . In the reveal phase, the argument system verifier simulates the PCP verifier to determine which symbols of π need to be examined (call the locations that the PCP verifier queries q_1, \dots, q_k). The verifier sends q_1, \dots, q_k to the prover to \mathcal{P} , and the prover sends back the answers $\pi(q_1), \dots, \pi(q_k)$, along with their authentication paths.

Completeness can be argued as follows. If the PCP satisfies perfect completeness, then whenever there exists a w such that $C(x, w) = y$, there is always some proof π that would convince the PCP verifier to accept. Hence, if the prover commits to π in the argument system, and executes the reveal phase as prescribed, the argument system verifier will also be convinced to accept.

Soundness can be argued roughly as follows. The analysis of Section 6.3.2.2 showed that the use of the Merkle tree binds the prover to a fixed string π' , in the sense that after the commit phase, for each possible query q_i , there is at most one value $\pi'(q_i)$ that the prover can successfully reveal without finding a collision under the hash function used to build the Merkle tree (and collision-finding is assumed to be intractable). Hence, if the argument system prover convinces the argument system verifier to accept, π' would convince the PCP verifier to accept. Soundness of the argument system is then immediate from soundness of the PCP system.

Remark 8.4. In order to turn a PCP into a succinct argument, we used a Merkle tree, and did not need to use a low-degree test. This is in contrast to Section 6.3, where we turned an interactive proof into a succinct argument by using a polynomial commitment scheme; the polynomial commitment scheme given in Section 6.3 combined a Merkle tree and a low-degree test.

However, the PCP approach to building succinct arguments has not “really” gotten rid of the low-degree test. It has just pushed it out of the commitment scheme and “into” the PCP. That is, short PCPs are

⁸⁸In the non-interactive argument obtained by applying the Fiat-Shamir transformation to Kilian's 4-message argument, the honest prover uses the random oracle in place of a collision-resistant hash function to build the Merkle tree over the PCP proof, and the PCP verifier's random coins are chosen by querying the random oracle at the root hash of the Merkle tree.

themselves typically based on low-degree polynomials, and the PCP itself typically makes use of a low-degree test.

A difference between the low-degree tests that normally go into short PCPs and the low-degree tests we’ve already seen is that short PCPs are usually based on low-degree *univariate* polynomials (see Section 8.4 for details). So the low-degree tests that go into short PCPs are targeted at univariate rather than multi-variate polynomials. Low-degree univariate polynomials are codewords in the Reed-Solomon error-correcting code, which is why many papers on PCPs refer to “Reed-Solomon PCPs” and “Reed-Solomon testing”. In contrast, efficient interactive proofs and MIPs are typically based on low-degree multivariate polynomials (also known as Reed-Muller codes), and hence use low-degree tests that are tailored to the multivariate setting.

8.2.1 Knowledge Soundness of Kilian and Micali’s Arguments

Recall (see Section 6.3.3) that an argument system satisfies knowledge soundness if, for any efficient prover \mathcal{P} that convinces the argument system verifier to accept with non-negligible probability, \mathcal{P} must *know* a witness w to the claim being proven. This is formalized by demanding that there is an efficient algorithm \mathcal{E} that is capable of outputting a valid witness if given the ability to repeatedly “run” \mathcal{P} .

Barak and Goldreich [BG02] showed that Kilian’s argument system is not only sound, but in fact knowledge-sound. (This assertion assumes that the underlying PCP that the argument system is based on also satisfies an analogous knowledge-soundness property, meaning that given a convincing PCP proof π , one can efficiently compute a witness. All of the PCPs that we cover in this survey have this knowledge-soundness property).

Valiant [Val08] furthermore showed that applying the Fiat-Shamir transformation to render Kilian’s argument system non-interactive (as per Micali [Mic00]) yields a *knowledge-sound* argument in the random oracle model. The rough idea of Valiant’s analysis is to show that, if a prover \mathcal{P} in the Fiat-Shamir-ed protocol produces an accepting transcript for Kilian’s interactive protocol, then one of following three events must have occurred: either (1) \mathcal{P} found a “hash collision” enabling it to break binding of the Merkle tree, or (2) \mathcal{P} built Merkle trees over one or more “unconvincing” PCP proofs π , yet hashing the Merkle-tree root produced randomness for the PCP verifier \mathcal{V} that caused \mathcal{V} to accept π anyway, or (3) \mathcal{P} built a Merkle tree over a “convincing” PCP proof π , and the first message of the transcript produced by \mathcal{P} is the root hash of this Merkle tree.

The first event is unlikely to occur unless the prover makes a huge number of queries to the random oracle. This is because the probability of finding a collision after T queries to the random oracle is at most $T^2/2^\lambda$ where 2^λ is the output length of the random oracle. The second event is also unlikely to occur, assuming the soundness error ϵ of the PCP is negligible. Specifically, if the prover makes T queries to the random oracle, the probability event (2) occurs is at most $T \cdot \epsilon$.

This means that (3) must hold (unless \mathcal{P} makes superpolynomially many queries to the random oracle). That is, any prover \mathcal{P} for the non-interactive argument that produces accepting transcripts with non-negligible probability must build a Merkle tree over a convincing PCP proof π and produce a transcript whose first message is the root hash of the Merkle tree. In this case, one can identify the entire Merkle tree by observing \mathcal{P} ’s queries to the random oracle. For example, if v_0 denotes the root hash provided in the transcript, then one can learn the values v_1, v_2 of the children of the root in the Merkle tree by looking for the (unique) query (v_1, v_2) made by \mathcal{P} to the random oracle R satisfying $R(v_1, v_2) = v_0$. Then one can learn the values of the grandchildren of the root by looking for the (unique) random oracle queries (v_3, v_4) and (v_5, v_6) made by \mathcal{P} such that $R(v_3, v_4) = v_1$ and $R(v_5, v_6) = v_2$. And so on.

The values of the leaves of the Merkle tree are just the symbols of the convincing PCP proof π . By

assumption that the PCP system satisfies knowledge-soundness, one can efficiently extract a witness from π .

8.3 A First Polynomial Length PCP, From a MIP

In light of Lemma 8.2, it is reasonable to ask whether the MIP of Section 7.2 can be transformed into a PCP for arithmetic circuit satisfiability, of length polynomial in the circuit size S . The answer is yes, though the polynomial is quite large (at least S^3).

Suppose we are given an instance (\mathcal{C}, x, y) of arithmetic circuit satisfiability, where \mathcal{C} is defined over field \mathbb{F} . Recall that in the MIP of Section 7.2, the verifier used the first prover to apply the sum-check protocol to a certain $(3 \log S)$ -variate polynomial $h_{x,y,z}$ over \mathbb{F} , where S is the size of \mathcal{C} . This polynomial was itself derived from a polynomial Z , claimed to equal the multilinear extension of a correct transcript for (\mathcal{C}, x, y) . The MIP verifier used the second prover to apply the point-vs-line low-degree test to the $O(\log S)$ -variate polynomial Z , which required the verifier to send \mathcal{P}_2 a random line in $\mathbb{F}^{\log S}$ (such a line can be specified with $2 \log S$ field elements). In order to achieve a soundness error of, say, $1/\log(n)$, it was sufficient to work over a field \mathbb{F} of size at least $\log(S)^{c_0}$ for a sufficiently large constant $c_0 > 0$.⁸⁹

The total number of bits that the verifier sent to each prover in this MIP was $r_Q = \Theta(\log(S) \log |\mathbb{F}|)$, since the verifier had to send a field element for each variable over which $h_{x,y,z}$ was defined. If $|\mathbb{F}| = \Theta(\log(S)^c)$, then $r_Q = \Theta(\log(S) \log \log(S))$. Applying Lemma 8.2 and Remark 8.3 to transform this MIP into a PCP, we obtain a PCP of length $\tilde{O}(2^{r_Q}) = S^{O(\log \log S)}$. This is slightly superpolynomial in S . On the plus side, the verifier runs in time $O(n + \log S)$, which is linear in the size n of the input assuming $S < 2^n$.

However, by tweaking the parameters used within the MIP itself, we can reduce r_Q from $O(\log(S) \log \log(S))$ to $O(\log S)$. Recall that within the MIP, each gate in \mathcal{C} was assigned a *binary* label, and the MIP made use of functions add_i , mult_i , io , I , and W that take as input $O(\log S)$ binary variables representing the labels of one or more gates. The polynomial $h_{x,y,z}$ was then defined in terms of the *multilinear* extensions of these functions. This led to an efficient MIP, in which the provers' runtime was $O(S \log S)$. But by defining the polynomials to be over $\Omega(\log S)$ many variables, r_Q becomes slightly super logarithmic, resulting in a PCP of length superpolynomial in S . To rectify this, we must find a way to redefine the polynomials, such that they involve fewer than $\log S$ variables.

To this end, suppose we assign each gate in \mathcal{C} a label in base b instead of base 2. That is, each gate label will consist of $\log_b(S)$ digits, each in $\{0, 1, \dots, b-1\}$. Then we can redefine the functions add_i , mult_i , io , I , and W to take as input $O(\log_b(S))$ variables representing the b -ary labels of one or more gates. Observe that, the larger b is, the smaller the number of variables these functions are defined over.

We can then define $h_{x,y,z}$ exactly as in Section 7.2, except if $b > 2$ then higher-degree extensions of add_i , mult_i , io , I , and W must be used in the definition, rather than multilinear extensions. Specifically, these functions, when defined over domain $\{0, 1, \dots, b-1\}^v$ for the relevant value of v , each have a suitable extension of degree at most b in each variable.

Compared to the MIP of Section 7.2, the use of the higher-degree extensions increases the degrees of all of the polynomials exchanged in the sum-check protocol and in the low-degree test by an $O(b)$ factor. Nonetheless, the soundness error remains at most $O(b \cdot \log_b(S)/|\mathbb{F}|^c)$ for some constant $c > 0$. Recall that we would like to take b as large as possible, but this is in tension with the requirement to keep the soundness

⁸⁹In cryptographic applications, one would want soundness error $n^{-\omega(1)}$ rather than $1/\log n$. The soundness error of the PCP in this section could be improved to $n^{-\omega(1)}$ by repeating the PCP $O(\log n)$ times independently, and rejecting if the PCP verifier rejects in any one of the runs. Such repetition is expensive in practice, but the PCP of this section is presented for didactic reasons and not meant to be practical.

Communication	Queries	\mathcal{V} time	\mathcal{P} time
$\text{polylog}(S)$ bits	$O(\log S / \log \log S)$	$O(n + \text{polylog}(S))$	$\text{poly}(S)$

Table 8.1: Costs of PCP of Section 8.3 for arithmetic circuit satisfiability (obtained from the MIP of Section 7.2), when run on a circuit \mathcal{C} of size S . The stated bound on \mathcal{P} 's time assumes \mathcal{P} knows a witness w for \mathcal{C} .

error $o(1)$ when working over a field of size polylogarithmic in S . Fortunately, it can be checked that for $b \leq O(\log(S)/\log \log(S))$, then $b \cdot \log_b(S) \leq \text{polylog}(S)$, and hence the soundness error is still at most $\text{polylog}(S)/|\mathbb{F}|^c$. In conclusion, if we set b to be on the order of $\log(S)/\log \log(S)$, then as long as the MIP works over a field \mathbb{F} of size that is a sufficiently large polynomial in $\log(S)$, the soundness error of the MIP is still at most, say, $1/\log n$.

For simplicity, let us choose b such that $b^b = S$. This choice of b is in the interval $[b_1, 2b_1]$ where $b_1 = \log(S)/\log \log(S)$.⁹⁰ In this modified MIP, the total number of bits sent from the verifier to the provers is $r_Q = O(b \cdot \log |\mathbb{F}|) = O((\log(S)/\log \log(S)) \cdot \log \log S) = O(\log S)$. If we apply Lemma 8.2 and Remark 8.3 to this MIP, the resulting PCP length is $\tilde{O}(2^{r_Q}) \leq \text{poly}(S)$.

Unfortunately, when we write $r_Q = O(\log S)$, the constant hidden by the Big-Oh notation is at least 3. This is because $h_{x,y,z}$ is defined over $3 \log_b(S)$ variables, which is at least $3b$ when $b^b = S$, and applying the sum-check protocol to $h_{x,y,z}$ requires \mathcal{V} to send at least one field element per variable. Meanwhile, the field size must be at least $3 \log_b(S) \geq 3b$ to ensure non-trivial soundness. Hence, $2^{r_Q} \geq (3b)^{3b} \geq S^{3-o(1)}$. So while the proof length of the PCP is polynomial in S , it is a large polynomial in S .

Nonetheless, this yields a non-trivial result: a PCP for arithmetic circuit satisfiability in which the prover's runtime is $\text{poly}(S)$, the verifier's is $O(n)$, and the number of queries the verifier makes to the proof oracle is $O(\log(S)/\log \log(S))$. As the total communication cost of the MIP is at most $\text{polylog}(S)$, all of the answers to the verifier's queries can be communicated in $\text{polylog}(S)$ bits in total (i.e., the alphabet size of the PCP is $|\Sigma| \leq 2^{\text{polylog}(S)}$). Applying the PCP-to-argument compiler of Section 8.2 yields a succinct argument for arithmetic circuit satisfiability with a verifier that runs in time $O(n)$ and a prover that runs in time $\text{poly}(S)$.

Remark 8.5. To clarify, the use of labels in base $b = 2$ rather than base $b = \Theta(\log(S)/\log \log(S))$ is the superior choice in interactive settings such as IPs and MIPs. The reason is that binary labels allow the IP or MIP prover(s) to manipulate and send polynomials of degree $O(1)$ in each round, and this keeps the prover runtime and communication costs low. In the context of the GKR protocol (Section 4.6), the MIP described in Section 7.2, and other interactive protocols that appear in the literature [BC12], the difference between the two parameter settings is a factor of at least b , and sometimes b^2 , in prover runtime and the total communication costs.

To recap, we have obtained a PCP for arithmetic circuit satisfiability with a linear time verifier, and a prover who can generate the proof in time *polynomial* in the size of the circuit. But to get a PCP that has any hope of being practical, we really need the prover time to be very close to *linear* in the size of the circuit. Obtaining such PCPs is quite complicated and challenging. Indeed, researchers have not yet had success in building plausibly practical VC protocols based on “short” PCPs, by which we mean PCPs for circuit satisfiability whose length is close to linear in the size of the circuit. To mitigate the bottlenecks in known short PCP constructions, researchers have turned to the more general *interactive oracle proof* (IOP) model. The following section and chapter cover highlights from this line of work. Specifically, Section 8.4 sketches the construction of PCPs for arithmetic circuit satisfiability where the PCP can be generated in time

⁹⁰Indeed, $b_1^{b_1} \leq S$, while $(2b_1)^{2b_1} \geq S^{2-o(1)}$.

quasilinear in the size of the circuit. This construction remains impractical and is mainly included in this survey for historical context. Chapter 9 describes IOPs that come closer to practicality.

8.4 A PCP of Quasilinear Length for Arithmetic Circuit Satisfiability

We have just seen (Sections 8.1 and 8.3) that known MIPs can fairly directly yield a PCP of polynomial size for simulating a (non-deterministic) Random Access Machine (RAM) M , in which the verifier runs in time linear in the size of the input x to M . But the proof length is a (possibly quite large) polynomial in the runtime T of M , and the length of a proof is of course a lower bound on the time required to generate it. This section describes how to use techniques tailored specifically to the PCP model to reduce the PCP length to $T \cdot \text{polylog}(T)$, while maintaining a verifier runtime of $n \cdot \text{polylog}(T)$.

The PCP described here originates in work of Ben-Sasson and Sudan [BS08]. Their work gave a PCP of size $\tilde{O}(T)$ in which the verifier runs in time $\text{poly}(n)$ and makes only a polylogarithmic number of queries to the proof oracle. Subsequent work by Ben-Sasson et al. [BGH⁺05] reduced the verifier's time to $n \cdot \text{polylog}(T)$. Finally, Ben-Sasson et al. [BSCGT13b] showed how the prover can actually generate the PCP in $T \cdot \text{polylog}(T)$ time using FFT techniques, and provided various concrete optimizations and improved soundness analysis. This PCP system is fairly involved, so we elide some details in this survey, seeking only to convey the main ideas.

8.4.1 Step 1: Reduce to checking that a polynomial vanishes on designated a subspace

In Ben-Sasson and Sudan's PCP, the claim that $M(x) = y$ is first turned into an equivalent circuit satisfiability instance $\{\mathcal{C}, x, y\}$, and the prover (or more precisely, the proof string π) claims to be holding a low-degree extension Z of a correct transcript W for $\{\mathcal{C}, x, y\}$, just like in the MIP of Section 7.2. And just as in the MIP, the first step of Ben-Sasson and Sudan's PCP is to construct a polynomial $g_{x,y,Z}$ such that Z extends a correct transcript for $\{\mathcal{C}, x, y\}$ if and only if $g_{x,y,Z}(a) = 0$ for all a in a certain set H .

The details, however, are different and somewhat more involved than the construction in the MIP. We elide several of these details here, and focus on highlighting the primary similarities and differences between the constructions in the PCP and the MIP of Section 7.2.

Most importantly, in the PCP, $g_{x,y,Z}$ **is a univariate polynomial**. The PCP views a correct transcript as a univariate function $W: [S] \rightarrow \mathbb{F}$ rather than as a v -variate function (for $v = \log S$) mapping $\{0, 1\}^v$ to \mathbb{F} as in the MIP. Hence, any extension Z of W is a univariate polynomial, and $g_{x,y,Z}$ is defined to be a univariate polynomial too. (The reason for using univariate polynomials is that it allows the PCP to utilize low-degree testing techniques in Steps 2 and 3 below that are tailored to univariate rather than multivariate polynomials. It is not currently known how to obtain PCPs of quasilinear length based on multivariate techniques, where by quasilinear length, we mean quasilinear in T , the runtime of the RAM that the prover is supposed to execute). Note that even the lowest-degree extension Z of W may have degree $|S| - 1$, which is much larger than the degrees of the multivariate polynomials that we've used in previous sections, and $g_{x,y,Z}$ will inherit this degree.

The univariate nature of $g_{x,y,Z}$ forces several additional differences in its construction, compared to the $O(\log S)$ -variate polynomial used in the MIP. In particular, in the univariate setting, $g_{x,y,Z}$ is specifically defined over a field of characteristic 2.⁹¹ The structure of fields of characteristic 2 are exploited multiple

⁹¹The characteristic of a field \mathbb{F} is the smallest number n such that $\underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = 0$. If a \mathbb{F} has size p^k for prime p and integer $k > 0$, then its characteristic is p . In particular, any field of size equal to a power of 2 has characteristic 2. We denote the field of size 2^k as \mathbb{F}_{2^k} .

Communication	Queries	\mathcal{V} time	\mathcal{P} time
$\text{polylog}(S)$ bits	$\text{polylog}(S)$	$O(n \cdot \text{polylog}(S))$	$O(S \cdot \text{polylog}(S))$

Table 8.2: Costs of PCP from Section 8.4 when run on a non-deterministic circuit \mathcal{C} of size S . The PCP is due to Ben-Sasson and Sudan [BS08], as refined by Ben-Sasson et al. [BGH⁺05] and [BSCGT13b]. The stated bound on \mathcal{P} 's time assumes \mathcal{P} knows a witness w for \mathcal{C} .

times in the construction of $g_{x,y,Z}$ and in the PCP as a whole. For example:

- Let us briefly recall a key aspect of the transformation from Section 5.5 that turned a RAM M into an equivalent circuit satisfiability instance $\{\mathcal{C}, x, y\}$. De Bruijn graphs played a role in the construction of \mathcal{C} , where they were used to “re-sort” a purported trace of the execution of M from time order into memory order.

To ensure that the MIP or PCP verifier does not have to fully materialize \mathcal{C} (which is of size at least T , far larger than the verifier’s allowed runtime of $n \cdot \text{polylog}(T)$), it is essential that \mathcal{C} have an “algebraically regular” wiring pattern. In particular, in both the PCP of this section and the MIP of Section 7.2, it is important that \mathcal{C} 's wiring pattern be “capturable” by a low-degree polynomial that the verifier can quickly evaluate. This is essential for ensuring that the polynomial $g_{x,y,Z}$ used within the MIP or PCP satisfies the following two essential qualities: (1) the degree of $g_{x,y,Z}$ is not much larger than that of Z (2) the verifier can efficiently evaluate $g_{x,y,Z}(r)$ at any point r , if given Z 's values at a handful of points derived from r .

In Ben-Sasson and Sudan’s PCP, the construction of $g_{x,y,Z}$ exploits the fact that there is a way to assign labels from $\mathbb{F} = \mathbb{F}_{2^\ell}$ to nodes in a De Bruijn graph such that, for each node v , the labels of the neighbors of v are *affine* (i.e., degree 1) functions of the label of v . (Similar to Section 5.5, the reason this holds boils down to the fact that the neighbors of a node with label v are simple bit-shifts of v . When v is an element of \mathbb{F}_{2^ℓ} , a bit-shift of v is an affine function of v .)

This is crucial for ensuring that the degree of $g_{x,y,Z}$ is not much larger than the degree of Z itself. In particular, the $g_{x,y,Z}: \mathbb{F} \rightarrow \mathbb{F}$ used in the PCP has the form

$$g_{x,y,Z}(z) = A(z, Z(N_1(z)), \dots, Z(N_k(z))), \quad (8.1)$$

where $(N_1(z), \dots, N_k(z))$ denotes the neighbors of node z in the De Bruijn graph, and A is a certain “constraint polynomial” of polylogarithmic degree. Since N_1, \dots, N_k are affine over \mathbb{F}_{2^ℓ} , $\deg(g_{x,y,Z})$ is at most a polylogarithmic factor larger than the degree of Z itself. Moreover, the verifier can efficiently evaluate each affine function N_1, \dots, N_k at a specified input r [BGH⁺05].

- The set H on which $g_{x,y,Z}$ should vanish if Z extends a correct transcript is chosen to ensure that the polynomial $\mathbb{Z}_H(z) = \prod_{\alpha \in H} (z - \alpha)$ is *sparse* (having $O(\text{polylog}(S))$ nonzero coefficients). The polynomial \mathbb{Z}_H is referred to as the *vanishing polynomial* for H , and via Lemma 8.3 in the next section, it plays a central role in the PCPs, IOPs (Chapter 9), and linear PCPs (Section 14.4) described hereafter in this survey. The sparsity of \mathbb{Z}_H ensures that it can be evaluated at any point in polylogarithmic time, even though H is a very large set (of size $\Omega(S)$). This will be crucial to allowing the verifier to run in polylogarithmic time in Step 2 of the PCP, discussed below. It turns out that if \mathbb{F} has characteristic $O(1)$ and H is a linear subspace of \mathbb{F} , then $\mathbb{Z}_H(z)$ has sparsity $O(\log S)$ as desired.

The final difference worth highlighting is that the field \mathbb{F}_{2^ℓ} over which $g_{x,y,Z}$ is defined must be small in the PCP, (or at least the set of inputs at which the verifier might query $g_{x,y,Z}$ must be a small). In particular,

the set must be of size $O(S \cdot \text{polylog}(S))$, since the proof length is lower bounded by the size of the set of inputs at which the verifier might ask for any evaluation of $g_{x,y,Z}$. This is in contrast to the MIP setting, where we were happy to work a very large field size (of size, say, 2^{128} or larger) to ensure negligible soundness error. This is a manifestation of the fact (mentioned in Section 8.1) that in a MIP the prover only has to “think up” answers to queries that the verifier actually asks, while in a PCP, the prover has to write down the answer to every possible query that the verifier *might* ask.

8.4.2 Step 2: Reduce to Checking that a Related Polynomial is Low-Degree

Note that checking whether a low-degree polynomial $g_{x,y,W}$ vanishes on H is very similar to the core statement checked in our MIP from Section 7.2. There, we checked that a *multilinear* polynomial derived from x, y , and W vanished on all Boolean inputs. Here, we are checking whether a *univariate* polynomial $g_{x,y,W}$ vanishes on all inputs in a pre-specified set H . We will rely on the following simple but essential lemma, which will arise several other times in this survey (including when we cover linear PCPs in Chapter 14).

Lemma 8.3. (*Ben-Sasson and Sudan [BS08]*) *A degree d univariate polynomial $g_{x,y,Z}(z)$ vanishes on H if and only if the polynomial $\mathbb{Z}_H(z) := \prod_{\alpha \in H} (z - \alpha)$ divides $g_{x,y,Z}(z)$, i.e., iff there exists a polynomial h^* with $\deg(h^*) \leq d - |H|$ such that $g_{x,y,Z}(z) = \mathbb{Z}_H(z) \cdot h^*(z)$.*

Proof. If $g_{x,y,W}(z) = \mathbb{Z}_H(z) \cdot h^*(z)$, then for any $\alpha \in H$, it holds that $g_{x,y,W}(\alpha) = \mathbb{Z}_H(\alpha) \cdot h^*(\alpha) = 0 \cdot \alpha = 0$, so $g_{x,y,W}$ indeed vanishes on H .

For the other direction, observe that if $g_{x,y,W}(\alpha) = 0$, then the polynomial $(z - \alpha)$ divides $g_{x,y,W}(z)$. It follows immediately that if $g_{x,y,W}$ vanishes on H , then $g_{x,y,W}$ is divisible by \mathbb{Z}_H . \square

So to convince \mathcal{V} that $g_{x,y,Z}$ vanishes on H , the proof merely needs to convince \mathcal{V} that $g_{x,y,Z}(z) = \mathbb{Z}_H(z) \cdot h^*(z)$ for some polynomial h^* of degree $d - |H|$. To be convinced of this, \mathcal{V} can pick a random point $r \in \mathbb{F}$ and check that

$$g_{x,y,Z}(r) = \mathbb{Z}_H(r) \cdot h^*(r). \tag{8.2}$$

Indeed, if $g_{x,y,Z} \neq \mathbb{Z}_H \cdot h^*$, then this equality will fail with probability $\frac{999}{1000}$ as long as $|\mathbb{F}|$ is at least 1000 times larger than the degrees of $g_{x,y,Z}$ and $\mathbb{Z}_H \cdot h^*$.

A PCP convincing \mathcal{V} that Equation (8.2) holds consists of four parts. The first part contains the evaluations of $Z(z)$ for all $z \in \mathbb{F}$. The second part contains a proof π_Z that Z has degree at most $|H| - 1$, and hence that $g_{x,y,Z}$ has degree at most $d = |H| \cdot \text{polylog}(S)$. The third part contains the evaluation of $h^*(z)$ for all $z \in \mathbb{F}$. The fourth part purportedly contains a proof π_{h^*} that $h^*(z)$ has degree at most $d - |H|$, and hence that $\mathbb{Z}_H \cdot h^*$ has degree at most d .

Let us assume that the verifier can efficiently check π_Z and π_{h^*} to confirm that Z and $h^*(z)$ have the claimed degrees (this will be the purpose of Step 3 below). \mathcal{V} can evaluate $g_{x,y,Z}(r)$ in quasilinear time after making a constant number of queries to the first part of the proof specifying Z . \mathcal{V} can compute $h^*(r)$ with a single query to the third part of the proof. Finally, \mathcal{V} can evaluate $\mathbb{Z}_H(r)$ without help in polylogarithmic time as described in Step 1 (Section 8.4.1). The verifier can then check that $g_{x,y,W}(r) = h^*(r) \cdot \mathbb{Z}_H(r)$.

In actuality, Step 3 will not be able to guarantee that π_Z and π_{h^*} are *exactly* equal to low-degree polynomials, but will be able to guarantee that, if the verifier’s checks all pass, then they are each close to some low-degree polynomial Y and h' respectively. One can then argue that $g_{x,y,Y}$ vanishes on H , analogously to the proof of Theorem 7.4 in the context of the MIP from Section 7.2.

8.4.3 Step 3: A PCP for Reed-Solomon Testing

Overview. The meat of the PCP construction is in this third step, which checks that a univariate polynomial has low-degree. This task is referred to in the literature as Reed-Solomon testing, because codewords in the Reed-Solomon error-correcting code consist of (evaluations of) low-degree univariate polynomials (cf. Remark 8.4).

The construction is recursive. The basic idea is to reduce the problem of checking that a *univariate* polynomial G_1 has degree at most d to the problem of checking that a related *bivariate* polynomial Q over \mathbb{F} has degree at most \sqrt{d} in each variable. It is known (cf. Lemma 8.5 below) how the latter problem can in turn be reduced back to a univariate problem, that is, to checking that a related univariate polynomial G_2 over \mathbb{F} has degree at most \sqrt{d} . Recursing $\ell = O(\log \log n)$ times results in checking that a polynomial G_ℓ has *constant* degree, which can be done with constantly many queries to the proof. We fill in some of the details of this outline below.

The precise soundness and completeness guarantees of this step are as follows. If G_1 indeed has degree at most d , then there is a proof π that is always accepted. Meanwhile, the soundness guarantee is that there is some universal constant k satisfying the following property: if a proof π is accepted with probability $1 - \varepsilon$, then there is a polynomial G of degree at most d such that G_1 agrees with G on at least a $1 - \varepsilon \cdot \log^k(S)$ fraction of points in \mathbb{F} (we say that G and G_1 are at most δ -far, for $\delta = \varepsilon \cdot \log^k(S)$.)

The claimed polylogarithmic query complexity of the PCP as a whole comes by repeating the base protocol, say, $m = \log^{2k}(S)$ times and rejecting if any run of the protocol ever rejects. If a proof π is accepted by the m -fold repetition with probability $1 - \varepsilon$, then it is accepted by the base protocol with probability at least $1 - \varepsilon / \log^k m$, implying that G is ε -far from a degree d polynomial G_1 .

Reducing Bivariate Low-Degree Testing on Product Sets to Univariate Testing. The bivariate low-degree testing technique described here is due to Spielman and Polishchuk [PS94]. Assume that Q is a bivariate polynomial defined on a product set $A \times B \subseteq \mathbb{F} \times \mathbb{F}$, claimed to have degree d in each variable. (In all recursive calls of the protocol, A and B will in fact both be subspaces of \mathbb{F} .) The goal is to reduce this claim to checking that a related univariate polynomial G_2 over \mathbb{F} has degree at most d .

Definition 8.4. For a set $U \subseteq \mathbb{F} \times \mathbb{F}$, partial bivariate function $Q: U \rightarrow \mathbb{F}$, and nonnegative integers d_1, d_2 , define $\delta^{d_1, d_2}(Q)$ to be the relative distance of Q from a polynomial of degree d_1 in its first variable and d_2 in its second variable.⁹² Formally,

$$\delta^{d_1, d_2}(Q) := \min_{f(x, y): U \rightarrow \mathbb{F}, \deg_x(f) \leq d_1, \deg_y(f) \leq d_2} \delta(Q, f).$$

Let $\delta^{d_1, *}(Q)$ and $\delta^{*, d_2}(Q)$ denote the relative distances when the degree in one of the variables is unrestricted.

Lemma 8.5. (Bivariate test on a product set [PS94]). *There exists a universal constant $c_0 \geq 1$ such that the following holds. For every $A, B \subseteq \mathbb{F}$ and integers $d_1 \leq |A|/4$, $d_2 \leq |B|/8$ and function $Q: A \times B \rightarrow \mathbb{F}$, it is the case that $\delta^{d_1, d_2}(Q) \leq c_0 \cdot (\delta^{d_1, *}(Q) + \delta^{*, d_2}(Q))$.*

The proof of Lemma 8.5 is not long, but we omit it from the survey for brevity.

Lemma 8.5 implies that, to test if a bivariate polynomial Q defined on a product set has degree at most d in each variable, it is sufficient to pick a variable $i \in \{1, 2\}$, then pick a random value $r \in \mathbb{F}$ and test whether the univariate polynomial $Q(r, \cdot)$ or $Q(\cdot, r)$ obtained by restricting the i th coordinate of Q to r has degree at most d .

⁹²By relative distance between Q and another polynomial P , we mean the fraction of inputs in $x \in U$ such that $Q(x) \neq P(x)$.

To be precise, if the above test passes with probability $1 - \varepsilon$, then $(\delta^{d,*}(Q) + \delta^{*,d}(f))/2 = \varepsilon$, and Lemma 8.5 implies that $\delta^{d,d}(Q) \leq 2 \cdot c_0 \cdot \varepsilon$. $Q(r, \cdot)$ and $Q(\cdot, r)$ are typically called a “random row” or “random column” of Q , respectively, and the above procedure is referred to as a “random row or column test”.

Note that $\delta^{d,d}(f)$ may be larger than the acceptance probability ε by only a constant factor $c_1 = 2c_0$. Ultimately, the PCP will recursively apply the “Reducing Bivariate Low-Degree Testing to Univariate Testing” technique $O(\log \log n)$ times, and each step may cause $\delta^{d_1, d_2}(Q)$ to blow up, relative to the rejection probability ε , by a factor of c_1 . This is why the final soundness guarantee states that, if the recursive test as a whole accepts a proof with probability $1 - \varepsilon$, then the input polynomial G_1 is δ -close to a degree d polynomial, where $\delta = \varepsilon \cdot c_1^{O(\log \log S)} \leq \varepsilon \cdot \text{polylog}(S)$.

Reducing Univariate Low-Degree Testing to Bivariate Testing on a Lower Degree Polynomial. Let G_1 be a univariate polynomial defined on a linear subspace L of \mathbb{F} (in all recursive calls of the protocol, the domain of G_1 will indeed be a linear subspace L of \mathbb{F}). Our goal in this step is to reduce testing that G_1 has degree at most d to testing that a related bivariate polynomial Q has degree at most \sqrt{d} in each variable. It is okay to assume that the number of vectors in L is at most a constant factor larger than d , as this will be the case every time this step is applied.

Lemma 8.6. [BS08] *Given any pair of polynomials $G_1(z)$, $q(z)$, there exists a unique bivariate polynomial $Q(x, y)$ with $\deg_x(Q) < \deg(G_1)$ and $\deg_y(Q) \leq \lfloor \deg(G_1)/\deg(q) \rfloor$ such that $G_1(z) = Q(z, q(z))$.*

Proof. Apply polynomial long-division to divide $G_1(z)$ by $(y - q(z))$, where throughout the long-division procedure, terms are ordered first by their degree in z and then by their degree in y .⁹³ This yields a representation of $G_1(z)$ as:

$$G_1(z) = Q_0(z, y) \cdot (y - q(z)) + Q(z, y). \quad (8.3)$$

By the basic properties of division in this ring, $\deg_y(Q) \leq \lfloor \deg(G_1)/\deg(q) \rfloor$, and $\deg_z(Q) < \deg(q)$. To complete the proof, set $y = q(z)$ and notice that the first summand on the right-hand side of Equation (8.3) vanishes. \square

By Lemma 8.6, to establish that G_1 has degree at most d , it suffices for a PCP to establish that $G_1(z) = Q(z, q(z))$, where the degree of Q in each variable is at most \sqrt{d} . Thus, as a first (naive) attempt, the proof could specify Q 's value on all points in $L \times \mathbb{F}$. Then \mathcal{V} can check that $G_1(z) = Q(z, q(z))$, by picking a random $r \in L$ and checking that $G_1(r) = Q(r, q(r))$. If this check passes, it is safe for \mathcal{V} to believe that $G_1(z) = Q(z, q(z))$, as long as Q is indeed low-degree in each variable, and we have indeed reduced testing that G_1 has degree $\approx d$ to testing that Q has degree at most \sqrt{d} in each variable.

The problem with the naive attempt is that the proof has length $|L| \cdot |\mathbb{F}|$, which is far too large; we need a proof of length $\tilde{O}(|L|)$. A second attempt might be to have the proof specify Q 's value on all points in the set $\mathcal{T} := \{(z, q(z)) : z \in L\}$. This would allow \mathcal{V} to check that $G_1(z) = Q(z, q(z))$ by picking a random $r \in L$ and checking that $G_1(r) = Q(r, q(r))$. While this shortens the proof to an appropriate size, the problem is that \mathcal{T} is not a product set, so Lemma 8.5 cannot be applied to check that Q has low-degree in each variable.

⁹³Polynomial long division repeatedly divides the highest-degree term of the remainder polynomial by the highest-degree term of the divisor polynomial to determine a new term to add to the quotient, stopping when the remainder has lower degree than the divisor. See https://en.wikipedia.org/wiki/Polynomial_long_division for details of the univariate case. For division involving multivariate polynomials, the “term of highest degree” is not well-defined until we impose a total ordering on the degree of terms. Ordering terms by their degree in z and breaking ties by their degree in y ensures that the polynomial long division is guaranteed to output a representation satisfying the properties described immediately after Equation (8.3).

To get around this issue, Ben-Sasson and Sudan ingeniously choose the polynomial $q(z)$ in such a way that there is a set \mathcal{B} of points, of size $O(|L|)$, at which it suffices to specify Q 's values. Specifically, they choose $q(z) = \prod_{\alpha \in L_0} (z - \alpha)$, where L_0 is a linear subspace of L containing \sqrt{d} vectors. Then $q(z)$ is not just a polynomial of degree \sqrt{d} , it is also a *linear map* on L , with kernel equal to L_0 . This has the effect of ensuring that $q(z)$ takes on just $|L|/|L_0|$ distinct values, as z ranges over L .

Ben-Sasson and Sudan use this property to show that, although \mathcal{T} is not a product set, it is possible to add $O(L)$ additional points \mathcal{S} to \mathcal{T} to ensure that $\mathcal{B} := \mathcal{S} \cup \mathcal{T}$ contains within it a large subset that is product. So \mathcal{P} need only provide Q 's evaluation on the points in \mathcal{B} : since $\mathcal{T} \subseteq \mathcal{B}$, the verifier can check that $G_1(z) = Q(z, q(z))$ by picking a random $r \in L$ and checking that $G_1(r) = Q(r, q(r))$, and since there is a large product set within $\mathcal{S} \cup \mathcal{T}$, Lemma 8.5 can be applied.

Chapter 9

Interactive Oracle Proofs

9.1 IOPs: Definition and Relation to IPs and PCPs

The concrete costs of the PCP prover of the previous section are very large. In this section, we describe more efficient protocols that operate in a generalization of the PCP setting, called Interactive Oracle Proofs (IOPs). Introduced by [BCS16, RRR16], IOPs in fact generalize both PCPs and IPs. An IOP is an IP where in each round the verifier is not forced to read the prover’s entire message, but rather is given query access to the prover’s message. This enables the IOP verifier to run in time sublinear in the total proof length (i.e., the sum of the lengths of all the messages sent by the prover).

Ben-Sasson, Chiesa, and Spooner [BCS16] showed that any IOP can be transformed into non-interactive argument in the random oracle model using Merkle-hashing and the Fiat-Shamir transformation, in a manner entirely analogous to the Kilian-Micali transformation from PCPs to succinct arguments of Section 8.2. Specifically, rather than sending the IOP prover’s message in each round of the IOP, the argument system prover sends a Merkle-commitment to the IOP prover’s message. The argument system verifier then simulates the IOP verifier to determine which elements of the message to query, and the argument system prover reveals the relevant symbols of the message by providing authentication paths in the Merkle tree. The interactive argument is then rendered non-interactive using the Fiat-Shamir transformation. As discussed in Section 4.7.2 when we first introduced the Fiat-Shamir transformation, if the IOP satisfies a property called round-by-round soundness, then the resulting non-interactive argument is sound in the random oracle model. All of the IOPs we discuss satisfy round-by-round soundness. (In more detail, Ben-Sasson et al. [BCS16] showed that the soundness of the non-interactive argument is *characterized* by a property of the IOP called soundness against state restoration attacks, and this property is in turn implied by round-by-round soundness [CCH⁺19]).

In this chapter, we give an IOP for R1CS-satisfiability that is concretely much more efficient for the prover than the PCP of the previous chapter (though unlike in Section 8.4.1, we will not endeavor to show that arbitrary computer programs have efficient reductions to R1CS-instances for which the IOP of this section avoids a pre-processing phase for the verifier). The IOP is comprised of two sub-protocols. The first is an efficient IOP for the task of reducing any R1CS-satisfiability instance to checking that a univariate polynomial is low degree. This is essentially a drop-in replacement for Sections 8.4.1 and 8.4.2, exploiting the ability of IOPs (unlike PCPs) to leverage interaction. The second is an IOP for Reed-Solomon testing called FRI (short for Fast Reed-Solomon Interactive Oracle Proof), in which the prover is more efficient than in the PCP of Section 8.4.3 for the same task.

Finally, in the last section of this chapter, we give an alternative IOP-based polynomial commitment

scheme that is implicit in work of Ames et al. [AHIV17] (Section 9.6). Roughly speaking, this polynomial commitment has significantly higher verifier complexity and query complexity than FRI, but is simpler and enjoys superior prover space complexity.

9.2 Background on FRI

9.2.1 Overview

FRI was introduced by Ben-Sasson, Bentov, Horesh, and Riabzev [BSBHR18], and its analysis has been improved over a sequence of works [BKS18a, SGKS20, BCI⁺20]. While we defer details of how FRI works until Section 9.4, it is useful now to precisely state the guarantee that it provides. Let d be a specified degree bound. The prover’s first message in FRI specifies a function $g: L_0 \rightarrow \mathbb{F}$, where L_0 is carefully chosen subset of \mathbb{F} of size $\rho^{-1} \cdot d$. Here, $0 < \rho < 1$ is a specified constant that is referred to as the *rate* of the Reed-Solomon code that the FRI prover claims g is a codeword in. In practice, protocols that use FRI choose \mathbb{F} to be significantly bigger than L_0 , because the message size (and hence prover runtime) is lower-bounded by $|L_0|$ (hence $|L_0|$ should be kept as small as possible) yet $|\mathbb{F}|$ should be large to ensure a strong soundness guarantee.

For specified parameter $\delta \in (0, 1 - \sqrt{\rho})$, known analyses of FRI guarantee that if the verifier accepts, then with overwhelming probability (say, at least $1 - 2^{-128}$), g is within relative distance δ of some polynomial p of degree at most d . That is, the number of points $r \in L_0$ for which $g(r) \neq p(r)$ is at most $\delta \cdot |L_0|$.

The query complexity of FRI is the dominant factor determining the proof length in succinct argument systems derived thereof, as each IOP query translates into a Merkle-tree authentication path that must be sent in the resulting argument system (see Section 8.2). Meanwhile, the prover runtime in FRI is mainly determined by the rate parameter ρ . This is because the smaller ρ is chosen, the longer the prover’s messages in the IOP, and hence the bigger the prover runtime to generate those messages. However, we will see that smaller choices of ρ potentially permit the FRI verifier to make fewer queries for a given security level, and hence keeps the proof shorter when the IOP is ultimately converted into an argument system. Argument system designers must therefore carefully choose ρ to obtain their preferred tradeoff between prover time and proof size.

9.2.2 Polynomial commitments and queries to points outside of L_0

We highlight the following subtlety of FRI. Many IOPs for circuit- or R1CS-satisfiability naturally demand the functionality of a polynomial commitment scheme (see Section 6.3.1). That is, the prover in the IOP must somehow send or commit to a low-degree polynomial p and the verifier must be able to force the prover to later evaluate the committed polynomial p at any point $r \in \mathbb{F}$ of the verifier’s choosing.

A natural attempt to use FRI to achieve this functionality is the following. To commit to a degree d polynomial p , the prover would send a function g (claimed to equal p) by specifying g ’s values over L_0 (a strict subset of \mathbb{F}). And the verifier can run FRI to confirm that (with overwhelming probability) g has relative distance at most δ from some degree d polynomial p . Note that if $\delta < \frac{1-d/|L_0|}{2} = \frac{1-\rho}{2}$, then p is unique, i.e., there can be only one degree d polynomial within relative distance δ of g . This is because any two distinct polynomials of degree at most d can agree on at most d points (Fact 2.1).

But since g is only specified via its evaluations at inputs in L_0 , how can the verifier determine evaluations of p on inputs $r \in \mathbb{F} \setminus L_0$? The research literature posits two approaches to dealing with this issue. The first is to carefully design IOPs for circuit- and R1CS-satisfiability, so that the verifier need not ever evaluate a polynomial specified by the prover at an input outside of L_0 . The second approach utilizes an observation

that will recur later in this survey when we cover pairing-based polynomial commitment schemes (Section 13.2.2). Specifically, for any degree- d univariate polynomial p , the assertion “ $p(r) = v$ ” is equivalent to the assertion that there exists a polynomial w of degree at most $d - 1$ such that

$$p(X) - v = w(X) \cdot (X - r). \quad (9.1)$$

This is a special case of Lemma 8.3.

As observed in [VP19], the above observation means that in order to confirm that $p(r) = v$, the verifier can apply FRI to the function $X \mapsto (g(X) - v) \cdot (X - r)^{-1}$ using degree bound $d - 1$ (we define this function to be 0 at input r). Note that whenever the FRI verifier queries this function at a point in L_0 , the evaluation can be obtained with one query to g at the same point. If the FRI verifier accepts, then with overwhelming probability this function is within distance δ of some polynomial q of degree at most $d - 1$. Since g and p have relative distance at most δ over domain L_0 , this means that the polynomials $q(X)(X - r)$ and $p(X)$ agree on at least $(1 - 2\delta) \cdot |L_0|$ inputs in L_0 , and both have degree at most d .

Suppose that $\delta < \frac{1-\rho}{2}$, which guarantees that $(1 - 2\delta)|L_0| > d$. Since any two distinct polynomials of degree at most d can agree on at most d inputs, this implies that $q(X) \cdot (X - r)$ and $p(X)$ are the same polynomial, and this in turn implies by Equation (9.1) that $p(r) = v$.

In summary, if the prover sends a function $g: L_0 \rightarrow \mathbb{F}$ and convinces the FRI verifier that g has distance at most $\delta < \frac{1-\rho}{2}$ from some degree d polynomial p , and moreover the FRI verifier accepts when applied to $X \mapsto (g(X) - v) \cdot (X - r)^{-1}$ using degree bound $d - 1$, then with overwhelming probability, $p(r)$ indeed equals v . That is, the verifier can safely accept that the low-degree polynomial p committed to via g evaluates to v at input r .

Note that the prover in this polynomial commitment scheme is bound to an actual polynomial p of degree at most d , in the sense that the prover must answer any evaluation request at input $r \in \mathbb{F}$ with $p(r)$ in order to convince the verifier to accept with non-negligible probability. This is in contrast to FRI by itself, which only binds the prover to a function g over domain L_0 that is *close to* p .

9.2.3 Costs of FRI

Prover time. In applications of FRI, the prover will know the coefficients of a degree- d polynomial p or its evaluations on a size- d subset of L_0 . To apply FRI to p , the prover must evaluate p at the remaining points in L_0 . This is the dominant cost in terms of prover runtime. The fastest known algorithms for this are essentially FFTs, and they require $\Theta(|L_0| \cdot \log |L_0|) = \Theta(\rho^{-1} d \log(\rho^{-1} d))$ field operations. For constant rate parameters ρ , this is $\Theta(d \log d)$ time. We remark that Ben-Sasson et al. [BSBHR18] describe the prover time in FRI as $O(d)$ field operations, but this assumes that the prover already knows the evaluations of p at all inputs in L_0 , which will not be the case in applications of FRI.

Proof length. FRI can be broken into two phases: a *commitment phase* and a *query phase*. The commitment phase is when the prover sends all of its messages in the IOP phase (during this phase, the verifier need not actually query any of the prover’s messages) and the verifier sends one random challenge to the prover in each of $\log_2 |L_0|$ rounds.

The query phase is when the verifier actually queries the prover’s messages at the necessary points to check the prover’s claims. The query phase in turn consists of a “basic protocol” that must be repeated many times to ensure good soundness error. Specifically, in the basic query protocol the verifier makes 2 queries to each each of the $\log_2 |L_0|$ messages sent by the prover. To ensure a $2^{-\lambda}$ upper bound on the probability that the FRI verifier accepts a function g of relative distance more than δ from any degree- d polynomial, the

basic protocol must be repeated roughly $\lambda / \log_2(1/(1-\delta))$ times. This query phase is the dominant cost in the proof length of the argument systems obtained by combining FRI-based IOPs with Merkle-hashing. The argument system prover must send a Merkle-tree authentication path (consisting of $O(\log d)$ hash values) for each query in the IOP, the proof length of the resulting arguments is $O(\lambda \cdot \log^2(d) / \log_2(1/(1-\delta)))$ hash values. For constant values of δ , this is $O(\lambda \cdot \log^2(d))$ hash values.

Remark 9.1. The FRI proof system is largely independent of the setting of the parameter δ . The only reason that the prover and verifier within FRI need to “know” what δ will be set to in the soundness analysis is to ensure that they repeat the query phase of FRI at least $\lambda / \log_2(1/(1-\delta))$ times.

9.3 An IOP for R1CS-SAT

9.3.1 The univariate sum-check protocol

The key technical fact exploited in this section relates the sum of any low-degree polynomial over a potentially large subset of inputs H to the polynomial’s evaluation at a *single* input, namely 0. Below, a non-empty subset $H \subseteq \mathbb{F}$ is said to be a multiplicative subgroup of field \mathbb{F} if H is closed under multiplication and inverses, i.e., for any $a, b \in H$, $a \cdot b \in H$, and $a^{-1}, b^{-1} \in H$.

Fact 9.1. *Let \mathbb{F} be a finite field and suppose that H is a multiplicative subgroup of \mathbb{F} of size n . Then for any polynomial q of degree less than $|H| = n$, $\sum_{a \in H} q(a) = q(0) \cdot |H|$. It follows that $\sum_{a \in H} q(a)$ is 0 if and only if $q(0) = 0$.*

We provide a proof of this fact. Our proof assumes several basic results in group theory, and may be safely skipped with no loss of continuity.

Proof. When H is a multiplicative subgroup of order n , it follows from Lagrange’s Theorem in group theory that $a^n = 1$ for any $a \in H$. Hence, H is precisely the set of n roots of the polynomial $X^n - 1$, i.e.,

$$\prod_{a \in H} (X - a) = X^n - 1. \quad (9.2)$$

We begin by proving the fact for $q(X) = X$, i.e., we show that $\sum_{a \in H} a = 0$. Indeed, it is easily seen that the coefficient of X^{n-1} when expanding out the left hand side of Equation (9.2) equals $-\sum_{a \in H} a$, and this must equal 0 because the coefficient of X^{n-1} on the right hand side of (9.2) is 0.

Now let $q(X)$ be any monomial $X \mapsto X^m$ for $1 < m < n$. It is known that any multiplicative subgroup of a finite field \mathbb{F} is cyclic, meaning there is some generator h such that $H = \{h, h^2, \dots, h^n\}$. Then

$$\sum_{a \in H} q(a) = \sum_{a \in H} a^m = \sum_{j=1}^n h^{m \cdot j}. \quad (9.3)$$

Another application of Lagrange’s theorem implies that if m and n are coprime, then h^m is also a generator of H , and hence $\sum_{j=1}^n h^{m \cdot j} = \sum_{j=1}^n h^j = 0$.

If m and n are not coprime, then it is known that the order of h^m is $d := \gcd(m, n)$, and hence letting $H' := \{h^m, h^{2m}, \dots, h^{(n/d)m}\}$, H is the disjoint union of the sets $H', h \cdot H', h^2 \cdot H', \dots$, and $h^{d-1} \cdot H'$, where for any $a \in \mathbb{F}$, $a \cdot H'$ denotes the set $\{a \cdot b : b \in H'\}$.

Since H' is a multiplicative subgroup of order n/d , the reasoning in the first paragraph of the proof shows that $\sum_{a \in H'} a = 0$. Hence, the right hand side of Equation (9.3) equals $(1 + h + h^2 + \dots + h^{d-1}) \cdot \sum_{a \in H'} a = 0$.

The lemma now follows for general polynomials $q(X) = \sum_{i=1}^{n-1} c_i X^i$ by linearity, combined with the obvious fact that for any constant $c \in \mathbb{F}$, $\sum_{a \in H} c = |H| \cdot c$. \square

For the remainder of this section, let H be a multiplicative subgroup of \mathbb{F} as in Fact 9.1. Let p be any univariate polynomial of degree at most D , where D may be greater than $|H|$. Recall that

$$\mathbb{Z}_H(X) = \prod_{a \in H} (X - a)$$

denotes the vanishing polynomial of H . Note that Equation (9.2) implies that $\mathbb{Z}_H(X) = X^n - 1$, i.e., \mathbb{Z}_H is sparse. We derive the following simple consequence of Fact 9.1.

Lemma 9.2. $\sum_{a \in H} p(a) = 0$ if and only if there exists polynomials h^*, f with $\deg(h^*) \leq D - n$ and $\deg(f) < n - 1$ satisfying:

$$p(X) = h^*(X) \cdot \mathbb{Z}_H(X) + X \cdot f(X). \quad (9.4)$$

Proof. Suppose first that Equation (9.4) holds. Then clearly

$$\sum_{a \in H} p(a) = \sum_{a \in H} (h^*(a) \cdot \mathbb{Z}_H(a) + a \cdot f(a)) = \sum_{a \in H} (h^*(a) \cdot 0 + a \cdot f(a)) = \sum_{a \in H} a \cdot f(a) = 0,$$

where the final equality holds by Fact 9.1 and the fact that $X \cdot f(X)$ evaluates to 0 on input 0.

Conversely, suppose that $\sum_{a \in H} p(a) = 0$. Dividing p by \mathbb{Z}_H allows us to write $p(X) = h^*(X) \cdot \mathbb{Z}_H(x) + r(X)$ for some remainder polynomial r of degree less than n . Since $0 = \sum_{a \in H} p(a) = \sum_{a \in H} r(a)$, we conclude by Fact 9.1 that r has no constant term. That is, we can write $r(X)$ as $X \cdot f(X)$ for some f of degree less than $n - 1$. \square

Lemma 9.2 offers a variant of the sum-check protocol of Section 4.1 that is tailored to compute sums of evaluations of univariate polynomials over multiplicative subgroup H rather than sums of multivariate polynomial evaluations over the Boolean hypercube. Specifically, in order to prove that a specified univariate polynomial p of degree D sums to 0 over a multiplicative subgroup H with $|H| = n$, Lemma 9.2 implies that it is sufficient for a prover to establish that there exists functions h^* and f such that f is a polynomial of degree at most $n - 1$ and h^* and f satisfy Equation (9.4).

The natural way to accomplish this is to have the prover send a message specifying f and h^* , use FRI to enable the verifier to confirm that both f and h^* are (close to) polynomials of the appropriate degree, and then confirm (with high probability) that Equation (9.4) holds by evaluating the left hand side and right hand side of Equation (9.4) at a random point $r \in \mathbb{F}$ and checking that the two evaluations are equal. This approach requires evaluating the polynomials f and h^* at an arbitrary point $r \in \mathbb{F}$, which runs into the subtlety described in Section 9.2.2 that r is not necessarily in the set L_0 over which the prover specifies f and h^* . While Section 9.2.2 describes a method that addresses this subtlety, below we describe an alternative approach (taken, e.g., in [COS20]). This sidesteps the issue by ensuring that the verifier need not send f at all, and need not evaluate h^* outside of L_0 . The reason that we cover this alternative approach is that it has concrete (but not asymptotic) efficiency benefits—for example, it avoids the need for the prover to explicitly send the polynomial f to the verifier.

Details of the univariate sum-check protocol. Recall that the prover’s goal is to establish that there exists functions h^* and f such that f is a polynomial of degree at most $n - 1$ and h^* and f satisfy Equation (9.4). To accomplish this, the prover will send a message specifying h^* , and the verifier will use FRI to confirm (with overwhelming probability) that h^* has relative distance at most δ from some degree- $(D - n)$ polynomial.

Observe that Equation (9.4) holds if and only if the function

$$f(X) := (p(X) - h^*(X)\mathbb{Z}_H(X)) \cdot X^{-1}$$

has degree at most $n - 1$ (while the above expression is undefined at $X = 0$; we simply define $f(0)$ to be 0). Hence, the verifier applies FRI to f to confirm that f has relative distance at most δ from some polynomial q of degree at most $n - 1$. Note that every time the FRI verifier needs to query f at a point $r \in L_0$, it can obtain f ’s value with one query to p and one query to h^* , as $f(r) = (p(r) - h^*(r)\mathbb{Z}_H(r)) \cdot r^{-1}$ (the verifier can evaluate $\mathbb{Z}_H(r)$ directly in $O(\log n)$ time).

If f has relative distance at most δ from q over domain L_0 , and h^* has relative distance at most δ from a degree- $(D - n)$ polynomial t over domain L_0 , then by definition of f , then $q(X) \cdot X$ and $p(X) - t(X)\mathbb{Z}_H(X)$ disagree on at most $2\delta \cdot |L_0|$ inputs in L_0 .

Since $q(X) \cdot X$ and $p(X) - t(X)\mathbb{Z}_H(X)$ are both polynomials of degree at most D , they cannot disagree on fewer than $|L_0| - D$ inputs unless they are the *same* polynomial. Hence, so long as $\delta < (1 - \rho)/2$,

$$q(X) \cdot X$$

and

$$p(X) - t(X)\mathbb{Z}_H(X)$$

must be the same polynomial. By Equation (9.4) (with $f = q$ and $h^* = t$), this implies that p sums to 0 over H .

Remark 9.2. It is also possible to give an analogous IOP for confirming that p sums to 0 over an *additive* rather than multiplicative subgroup H of \mathbb{F} . This can be useful when working over fields of characteristic 2 (i.e., of size equal to a power of 2), since if a field has size 2^k for positive integer k , then it has an additive subgroup H of size $2^{k'}$ for every positive integer $k' < k$; moreover the vanishing polynomial $\mathbb{Z}_H(Y) = \prod_{a \in H} (a - Y)$ is sparse (just as in the PCP of Section 8.4.1). Similarly, FRI (Section 9.4) can also be instantiated over additive rather than multiplicative subgroups.

9.3.2 An IOP for R1CS-SAT via Univariate Sum-Check

In this section we explain why the ability to solve the univariate sum-check problem suffices to give an IOP for R1CS-SAT. The reader may wonder, since IOPs are able to leverage interaction, why not just use the same techniques as in the MIP for R1CS-SAT of Section 7.5.2? The answer is that the MIP worked with multilinear polynomials over $O(\log n)$ -variables, leading to logarithmically many rounds of interaction. Here, we are seeking to only have the prover commit to univariate polynomials, because the FRI IOP for Reed-Solomon testing of Section 9.4 only applies to univariate polynomials^{94,95} This means that in this

⁹⁴Though we nonetheless explain in Section 9.5 how to build upon any Reed-Solomon testing IOP in an indirect manner to obtain a polynomial commitment multilinear polynomials in the IOP model.

⁹⁵An added benefit of using univariate and bivariate polynomials is that the IOP will be $O(1)$ rounds (ignoring FRI itself, which requires logarithmically many rounds). In later chapters, we consider replacing FRI with the pairing-based polynomial commitment schemes of Section 13.2.2, achieving constant proof length. This proof length would not be possible if the IOP used super-constantly many rounds.

section we have to “redo” the MIP, replacing each constituent multilinear polynomial appearing in that protocol with a univariate or bivariate analog.

Recall from Section 7.5 that an RICS-SAT instance is specified by $m \times n$ matrices A, B, C , and the prover wishes to demonstrate that it knows a vector z such that $Az \circ Bz = Cz$, where \circ denotes Hadamard (entrywise) product. For simplicity, we will assume that $m = n$ and that there is a multiplicative subgroup H of \mathbb{F} of size exactly n . Here is how the prover can do this. Let us label the n entries of z with elements of H , and let \hat{z} be the unique univariate polynomial of degree at most n over \mathbb{F} that extends z in the sense that $\hat{z}(h) = z_h$ for all $h \in H$. Similarly, let $z_A = Az$, $z_B = Bz$, and $z_C = Cz$ be vectors in \mathbb{F}^n , and let $\hat{z}_A, \hat{z}_B, \hat{z}_C$ extend z_A, z_B, z_C . To check that indeed $Az \circ Bz = Cz$, the verifier must confirm two properties. First:

$$\text{for all } h \in H, \hat{z}_A(h) \cdot \hat{z}_B(h) = \hat{z}_C(h). \quad (9.5)$$

Second:

$$\text{for all } h \in H, \text{ and } M \in \{A, B, C\}, \hat{z}_M(h) = \sum_{j \in H} M_{h,j} \cdot \hat{z}(j). \quad (9.6)$$

Equation (9.6) ensures that z_A, z_B, z_C are indeed equal to Az, Bz , and Cz . Assuming this to be so, Equation (9.5) confirms that $Az \circ Bz = Cz$.

The IOP prover can commit to degree- n polynomials $\hat{z}, \hat{z}_A, \hat{z}_B$, and \hat{z}_C . We assume for the remainder of this section that all functions committed to by the prover are all in fact exactly equal polynomials of degree at most n , and that the verifier can request evaluations of these polynomials at any point in the field. This assumption is justified by Section 9.2.2, which shows how to use FRI to obtain a polynomial commitment scheme.⁹⁶

Checking Equation (9.5). By Lemma 8.3 from the previous chapter, the first check is equivalent to the existence of a polynomial h^* of degree at most n such that

$$\hat{z}_A(X) \cdot \hat{z}_B(X) - \hat{z}_C(X) = h^*(X) \cdot \mathbb{Z}_H(X). \quad (9.7)$$

The prover can commit to the polynomial h^* via Section 9.2.2. The verifier can probabilistically check that Equation (9.7) holds by choosing a random $r \in \mathbb{F}$ and confirming that

$$\hat{z}_A(r) \cdot \hat{z}_B(r) - \hat{z}_C(r) = h^*(r) \cdot \mathbb{Z}_H(r). \quad (9.8)$$

This requires querying the committed polynomials $\hat{z}_A, \hat{z}_B, \hat{z}_C$, and h^* at r ; the verifier can evaluate $\mathbb{Z}_H(r)$ on its own in logarithmic time because $\mathbb{Z}_H(r)$ is sparse. If all functions committed by the prover are polynomials of degree at most n , then up to soundness error $2n/|\mathbb{F}|$ over the choice of r , if Equation (9.8) holds at r then it is safe for the verifier to believe that Equation (9.7) holds, and hence also Equation (9.5).

Checking Equation (9.6). To check that Expression (9.6) holds, we leverage interaction, a resource that was not available to the PCP of Section 8.4. Fix $M \in \{A, B, C\}$ for the remainder of the paragraph. Let $\hat{M}(X, Y)$ denote the bivariate low-degree extension of the matrix M , interpreted in the natural manner as a function $M(x, y): H \times H \rightarrow \mathbb{F}$ via $M(x, y) = M_{x,y}$. That is, $\hat{M}(x, y)$ is the unique bivariate polynomial of degree at most n in each variable that extends M . Since \hat{z}_M is the *unique* extension of z_M of degree less than

⁹⁶As with the univariate sum-check protocol itself (Section 9.3.1), concrete efficiency improvements can be had by avoiding the technique of Section 9.2.2 and instead having the verifier only evaluate the functions at points in L_0 . We do not take that approach here to keep the protocol description as simple as possible.

n , it is easily seen that Equation (9.6) holds for all $h \in H$ if and only if the following equality holds as formal polynomials:

$$\hat{z}_M(X) = \sum_{j \in H} \hat{M}(X, j) \hat{z}(j). \quad (9.9)$$

Since any two distinct polynomials of degree at most n can agree on at most n inputs, if the verifier chooses r' at random from \mathbb{F} , then up to soundness error $n/|\mathbb{F}|$ over the choice of r' , Equation (9.6) holds if and only if

$$\hat{z}_M(r') = \sum_{j \in H} \hat{M}(r', j) \hat{z}(j). \quad (9.10)$$

The verifier checks Equation (9.10) by sending r' to the prover and proceeding as follows. Let

$$q(Y) = \hat{M}(r', Y) \hat{z}(Y) - \hat{z}_M(r') \cdot |H|^{-1},$$

so that the validity of Equation (9.10) is equivalent to $\sum_{j \in H} q(Y) = 0$. The verifier requests that the prover establish that $\sum_{j \in H} q(Y) = 0$ by applying the univariate sum-check protocol discussed after Lemma 9.2.

At the end of the univariate sum-check protocol applied to q , the verifier needs to evaluate q at a randomly chosen point r'' . Clearly this can be done in a constant number of field operations if the verifier is given $\hat{z}(r)$, $\hat{z}_M(r)$, and $\hat{M}(r', r'')$. The first two evaluations, $\hat{z}(r)$ and $\hat{z}_M(r)$, can be obtained because the prover already committed to the polynomials \hat{z} and \hat{z}_M via the technique described in Section 9.2.2.

How the verifier computes $\hat{M}(r', r'')$. All that remains is to explain how and when the verifier can efficiently obtain $\hat{M}(r', r'')$. For some “structured” matrices M , it is possible that \hat{M} may be evaluateable in time polylogarithmic in n . This situation is analogous to how the verifier in the GKR protocol or the MIP of Section 7.2 avoids pre-processing so long as the multilinear extensions of the wiring predicates of the circuit- or RICS-satisfiability instance can be evaluated efficiently.

For unstructured matrices M , Chiesa et al. [CHM⁺19, COS20] give a technique through which a *trusted* party can, in pre-processing, commit to \hat{M} , and this permits the *untrusted* prover to efficiently and verifiably reveal $\hat{M}(r', r'')$ to the verifier as needed during the IOP just described. The goal will be for the commitment phase to take time proportional to the number K of *nonzero* entries of M , as opposed to the total number of entries in M (i.e., n^2). Indeed, in many applications K will be much smaller than n^2 (see for example Section 7.5.1). This is analogous to how we use the commitment scheme for sparse *multilinear polynomials* described later in this survey (Section 13.3) to commit to the sparse multilinear extensions add and mult of the wiring predicates used in the GKR protocol and its variants. In both cases, the general idea is to express the sparse polynomial \hat{M} to be committed in terms of a constant number of *dense* polynomials, each of which can be straightforwardly committed in time proportional to K .

The key to achieving this in the IOP setting is to give an explicit expression for \hat{M} , analogous to how Lemma 3.6 represents the multilinear extension of any function defined over the Boolean hypercube in terms of the Lagrange basis. Let us motivate the expression as follows. Recall that when designing IPs and MIPs, we defined $\tilde{\beta}$ to be the unique multilinear extension of the “equality function” that takes two inputs from the Boolean hypercube and outputs 1 if and only if they are equal (see for example Equation (4.17)). In the IOP setting of this section, the analog of the Boolean hypercube is the subgroup H , and the analog of $\tilde{\beta}$ is the following bivariate polynomial: $u_H(X, Y) := \frac{\mathbb{Z}_H(X) - \mathbb{Z}_H(Y)}{X - Y}$. Though it is not immediately obvious, u_H is

a polynomial of degree at most $|H| = n$ in each variable.⁹⁷ It is easy to check that for $x, y \in H$ with $x \neq y$, $u_H(x, y) = 0$. While less obvious, it is also true that for all $x \in H$, $u_H(x, x) \neq 0$ (though unlike $\tilde{\beta}$, it is not necessarily the case that $u_H(x, x) = 1$ for all $x \in H$).

Let \mathcal{K} be a multiplicative subgroup of \mathbb{F} of order K . Let us define three functions val , row , col mapping \mathcal{K} to \mathbb{F} as follows. We impose some canonical bijection between the nonzero entries of M and \mathcal{K} , and for $\kappa \in \mathcal{K}$, we define $row(\kappa)$ and $col(\kappa)$ to be the row index and column index of the κ 'th nonzero entry of M , and define $val(\kappa)$ to be the value of this entry, divided by $u_H(row(\kappa), row(\kappa)) \cdot u_H(col(\kappa), col(\kappa))$. Let \hat{val} , \hat{row} , and \hat{col} be their unique extensions of degree at most K . Then we can express

$$\hat{M}(X, Y) = \sum_{\kappa \in \mathcal{K}} u_H(X, \hat{row}(\kappa)) \cdot u_H(Y, \hat{col}(\kappa)) \cdot \hat{val}(\kappa). \quad (9.11)$$

Indeed, it is easy to see that the right hand side of the above equation has degree at most $|H|$ in both X and Y , and agrees with \hat{M} at all inputs in $H \times H$. Since \hat{M} is the unique polynomial with these properties, the right hand side and left hand side are the same polynomial.

Equation (9.11) expresses \hat{M} in terms of degree- κ polynomials \hat{row} , \hat{col} , and \hat{val} , which suggests the following approach to permitting the verifier to efficiently learn $\hat{M}(r', r'')$ at the end of the IOP. The pre-processing phase can have a trusted party commit to the polynomials \hat{val} , \hat{row} , \hat{col} (since these are degree- K polynomials, they can be specified in time proportional to K using the scheme of Section 9.2.2), and then when the verifier needs to know $\hat{M}(r', r'')$, the univariate sum-check protocol is invoked to establish that the polynomial

$$p(\kappa) := u_H(r', \hat{row}(\kappa)) \cdot u_H(r'', \hat{col}(\kappa)) \cdot \hat{val}(\kappa) \quad (9.12)$$

sums to the claimed value over inputs in \mathcal{K} .

This unfortunately does not yield the efficiency we desire, because $u_H(r', \hat{row}(\kappa))$ and $u_H(r'', \hat{col}(\kappa))$ have degree as large as $n \cdot K$, since u_H has degree n in both of its variables. This means that applying the univariate sum-check protocol to $p(\kappa)$ would require the IOP prover to send a polynomial h^* of degree $\Theta(n \cdot K)$, when we are seeking an IOP with prover runtime (and hence proof length) proportional just to K . To address this issue, let us define f to be the unique polynomial of degree at most K that agrees with p at all inputs in \mathcal{K} . We are going to have the prover commit to f , and in order for the verifier to be able to check that f and p agree at all inputs in \mathcal{K} , we will need to identify a new expression (simpler than Equation (9.12)) that describes p 's values at inputs in \mathcal{K} .

Specifically, observe that for any $a \in \mathcal{K}$,

$$u_H(r', \hat{row}(a)) = \frac{\mathbb{Z}_H(r') - \mathbb{Z}_H(\hat{row}(a))}{(r' - \hat{row}(a))} = \frac{\mathbb{Z}_H(r')}{(r' - \hat{row}(a))},$$

where the final equality uses the fact that $\hat{row}(a) \in H$. Similarly, for any $a \in \mathcal{K}$,

$$u_H(r'', \hat{col}(a)) = \frac{\mathbb{Z}_H(r'')}{(r'' - \hat{col}(a))}.$$

Hence, for any $a \in \mathcal{K}$,

$$p(a) = \frac{\mathbb{Z}_H(r') \mathbb{Z}_H(r'') \cdot \hat{val}(a)}{(r' - \hat{row}(a)) \cdot (r'' - \hat{col}(a))}. \quad (9.13)$$

⁹⁷To see that $p_1(X, Y) := \mathbb{Z}_H(X) - \mathbb{Z}_H(Y)$ is divisible by $p_2 := X - Y$, observe that standard properties of polynomial division imply that when p_1 is divided by p_2 , the remainder polynomial $r(X, Y)$ can be taken to have degree in X strictly less than that of p_2 in X , which is 1. Hence, $r(X, Y)$ has degree 0 in X . Since p_1 is symmetric, it can be seen that r is also symmetric, and hence $r(X, Y)$ is constant.

This discussion leads to the following protocol enabling the verifier to efficiently learn $\hat{M}(r', r'')$ following a pre-processing phase during which a trusted party commits to the degree- K polynomials $\hat{r}ow$, $\hat{c}ol$, and $\hat{v}al$. First, the prover commits to the degree- K polynomial f defined above, and the prover and verifier apply the univariate sum-check protocol to compute $\sum_{a \in \mathcal{K}} f(a)$ (recall that if f is as claimed, then this quantity equals $\hat{M}(r', r'')$).

Second, observe that for all $a \in \mathcal{K}$, $f(a)$ equals the expression in Equation (9.13) if and only if the following polynomial vanishes for all $a \in \mathcal{K}$:

$$(r' - \hat{r}ow(a)) \cdot (r'' - \hat{c}ol(a)) \cdot f - \mathbb{Z}_H(r') \mathbb{Z}_H(r'') \cdot \hat{v}al(a). \quad (9.14)$$

By Lemma 8.3, Expression (9.14) vanishes for all $a \in \mathcal{K}$ if and only if it is divisible by $\mathbb{Z}_{\mathcal{K}}(Y) = \prod_{a \in \mathcal{K}} (Y - a)$. The prover establishes this by committing to a polynomial q such that $q \cdot \mathbb{Z}_{\mathcal{K}}$ equals Expression (9.14), and the verifier checks the claimed polynomial equality by confirming that it holds at a random input $r''' \in \mathbb{F}$. This does require the verifier to evaluate $\hat{r}ow$, $\hat{c}ol$, $\hat{v}al$, f , q , and $\mathbb{Z}_{\mathcal{K}}$ at r''' ; the first three evaluations can be obtained from the pre-processing commitments to these polynomials, while $f(r''')$ and $q(r''')$ can be obtained from the prover's commitments to f and q , and $\mathbb{Z}_{\mathcal{K}}(r)$ can be computed in logarithmic time because it is sparse.

The polynomials that the prover commits to in the univariate sum-check protocol and in verifier's second check (namely, f and q) have degree at most $2K$, so the IOP proof length is $O(K)$.

Optimizations affecting concrete efficiency. [BSCR⁺19, CHM⁺19, COS20] contain a number of optimizations on the IOP of this section that improve concrete but not asymptotic efficiency. For example, rather than applying FRI to every function sent by the prover to ensure the functions are all low-degree, FRI is applied once, to a random linear combination of the functions. Each query that the FRI verifier makes to the random linear combination can be answered with one query to each of the constituent functions making up the linear combination. Lemma 9.3 in the next section implies that if even one of the functions sent by the prover is far from a low-degree polynomial, then the random linear combination will also be far from a low-degree polynomial, and hence the FRI verifier will reject with high probability. By only applying FRI once to the random linear combination rather than once for every constituent polynomial, the constant factors in the proof length, prover and verifier complexity, etc. are significantly reduced. See [COS20] for additional details.

9.4 Details of FRI: Better Reed-Solomon Proximity Proofs via Interaction

Recall that the PCP for Reed-Solomon testing sketched in Section 8.4.3 worked by iteratively reducing the problem of testing a function G_i for proximity to a degree d_i polynomial to the problem of testing a related function G_{i+1} for proximity to a degree d_{i+1} polynomial where $d_{i+1} \ll d_i$ (more precisely, $d_{i+1} \approx \sqrt{d_i}$). A source of inefficiency in this construction was that each iterative reduction incurred a constant-factor loss in the distance from any low-degree polynomial of the function being analyzed. That is, if G_i is at least δ_i -far from every degree d_i polynomial, then G_{i+1} is only guaranteed to be at least (δ_i/c_0) -far from every polynomial of degree at most $\sqrt{d_i}$ for some universal constant $c_0 > 1$. This constant-factor loss in distance per iteration meant that we had to keep the number of iterations small if we wanted to maintain meaningful soundness guarantees. This in turn meant we needed to make sure that we made a *lot* of progress in reducing the degree parameter in each iteration. This is why we choose for d_{i+1} to be just $\sqrt{d_i}$ —this ensured the d_i fell doubly-exponentially quickly in i , i.e., only $\Theta(\log \log d_0)$ iterations were required before the degree became 0.

Unlike PCPs, IOPs such as FRI are allowed to be interactive, and FRI exploits interaction to ensure that the distance parameter δ_i does *not* fall by a constant factor in each round. This permits FRI to use exponentially more iterations— $\Theta(\log d_0)$ rather than $\Theta(\log \log d_0)$ —while maintaining meaningful soundness guarantees, with corresponding efficiency benefits.

Recall from Section 9.2.1 that FRI proceeds in two phases, a *commitment phase* and a *query phase*. The commitment phase is when the prover sends all of its messages in the IOP phase (during this phase, the verifier need not actually query any of the prover’s messages) and the verifier sends one random challenge to the prover in each round. The query phase is when the verifier queries the prover’s messages at the necessary points to check the prover’s claims.

Comparison of the IOP Commitment Phase to Section 8.4.3. For simplicity, let us suppose that in round i of the IOP, G_i is a function defined over a multiplicative subgroup L_i of \mathbb{F} , where $|L_i|$ is a power of 2, and the current degree bound d_i is also a power of 2. In round 0, G_0 is the truth table of the polynomial defined over domain L_0 , for which we are testing proximity to univariate polynomials of degree d_0 .

Recall that in Section 8.4.3, to show that G_i was a degree d_i polynomial, for any desired polynomial q_i of degree $\sqrt{d_i}$, it sufficed for the PCP to establish that $G_i(z) = Q_i(z, q_i(z))$ for some bivariate polynomial Q_i of degree at most $\sqrt{d_i}$ in each variable. When G_i indeed has degree at most d_i , the existence of such a polynomial Q_i was guaranteed by Lemma 8.6.

In the IOP of this section, $q_i(z)$ will simply be z^2 (since this choice of q_i does not depend on i , we omit the subscript i from q henceforth). When G_i indeed has degree at most d_i , Lemma 8.6 guarantees that there is a $Q_i(X, Y)$ of degree at most 1 in X and at most $d_i/2$ in Y such that $Q_i(z, z^2) = G_i(z)$. Under this setting of $q_i(z)$, this representation of G_i has an especially simple form. Let $P_{i,0}$ (respectively, $P_{i,1}$) consist of all monomials of G_i of even (respectively, odd) degree, but with all powers divided by two and then replaced by their integer floor. For example, if $G_i(z) = z^3 + 3z^2 + 2z + 1$, then $P_{i,0} = 3z + 1$ and $P_{i,1}(z) = z + 2$. When $q(z) = z^2$, Lemma 8.6 is simply observing that we can ensure that $G_i(z) = Q_i(z, z^2)$ by defining $Q_i(z, y) := P_{i,0}(y) + z \cdot P_{i,1}(y)$.

In the PCP for Reed-Solomon testing of Section 8.4.3, $q(z)$ was chosen to be a polynomial of degree $\sqrt{d_i}$ such that the size of the image $q(L_i)$ was much smaller than $|L_i|$ itself (smaller by a factor of $\sqrt{d_i}$). Similarly, when L_i is a multiplicative subgroup of \mathbb{F} , the map $z \rightarrow z^2$ is two-to-one on L_i ,⁹⁸ so under our choice of $q(z) := z^2$, if we define

$$L_{i+1} = q(L_i), \tag{9.15}$$

then $|L_{i+1}| = |L_i|/2$.

Complete Description of the IOP Commitment Phase. After the IOP prover commits to the polynomial Q_i defined over domain L_i , the IOP verifier chooses a random value $x_i \in \mathbb{F}$ and requests that the prover send it the univariate polynomial

$$G_{i+1}(Y) := Q_i(x_i, Y) = P_{i,0}(Y) + x_i \cdot P_{i,1}(Y), \tag{9.16}$$

defined over the domain L_{i+1} given in Equation (9.15).

⁹⁸To see this, recall from the proof of Fact 9.1 that any multiplicative subgroup H of a finite field \mathbb{F} is cyclic, meaning there is a $h \in H$ such that $H = \{h, h^2, \dots, h^{|H|}\}$, where $h^{|H|} = 1$. If $|H|$ is even, this means that $H' := \{h^2, h^4, \dots, h^{|H|}\}$ is also a multiplicative subgroup of \mathbb{F} , of order $|H|/2$, and H' consists of all perfect squares (also known as quadratic residues) in H . For each element h^{2i} in H' , h^{2i} is the square of both h^i and $h^{i+|H|/2}$.

This proceeds for $i = 0, 1, \dots, \log_2(d_0)$. Finally for $i^* = \log_2(d_0)$, $G_{i^*}(Y)$ is supposed to have degree 0 and hence be a constant function. In this round, the prover's message simply specifies the constant C , which the verifier interprets to specify that $G_{i^*}(Y) = C$.

Query Phase. The verifier \mathcal{V} repeats the following ℓ times, for a parameter ℓ we set later. \mathcal{V} picks an input $s_0 \in L_0$ at random, and for $i = 0, \dots, i^* - 1$, \mathcal{V} sets $s_{i+1} = q(s_i) = s_i^2$. The verifier then wishes to check that $G_{i+1}(s_{i+1})$ is consistent with Equation (9.16) at input s_{i+1} , i.e., that $G_{i+1}(s_{i+1})$ indeed equals $Q_i(x_i, s_{i+1})$. We now explain how this check can be performed with two queries to G_i .

Observe that $g(X) = Q_i(X, s_{i+1})$ is a linear function in X , hence the entire function can be inferred from its evaluations at two inputs. Specifically, let $s'_i \neq s_i$ denote the other element of L_i satisfying $(s'_i)^2 = s_{i+1}$. We know that $g(s_i) = Q_i(s_i, s_{i+1}) = G_i(s_i)$, while $g(s'_i) = Q_i(s'_i, s_{i+1}) = G_i(s'_i)$. And since g is linear, these two evaluations are enough to infer the entire linear function g , and thereby evaluate $g(x_i)$. More concretely, it holds that

$$g(X) = (X - s_i) \cdot (s'_i - s_i)^{-1} \cdot G_i(s'_i) + (X - s'_i) \cdot (s_i - s'_i)^{-1} \cdot G_i(s_i),$$

as this expression is a linear function of X that takes the appropriate values at $X = s_i$ and $X = s'_i$.

Accordingly, to check that $G_{i+1}(s_{i+1})$ indeed equals $Q_i(x_i, s_{i+1})$, the verifier queries G_i at s'_i and s_i , and checks that

$$G_{i+1}(s_{i+1}) = (x_i - s_i) \cdot (s'_i - s_i)^{-1} \cdot G_i(s'_i) + (x_i - s'_i) \cdot (s_i - s'_i)^{-1} \cdot G_i(s_i). \quad (9.17)$$

Completeness and Soundness. Completeness of the protocol holds by design: it is clear that if G_0 is indeed a univariate polynomial of degree at most d_0 over domain L_0 and sends the prescribed messages, then all of the verifier's checks will pass. Indeed, all of the consistency checks will pass, and G_{i^*} will indeed be a constant function.

The state-of-the-art soundness guarantee for FRI is stated in Theorem 9.4 below. Its proof is quite technical and is omitted from the survey, but we sketch the main ideas in detail.

Worst-Case to Average-Case Reductions for Reed-Solomon Codes. The key technical notion in the analysis of FRI is the following statement. Let f_1, \dots, f_ℓ be a collection of ℓ functions on domain L_i , and suppose that at least one of f_j has relative distance at least δ from every polynomial of degree at most d_i over L_i . Then if $f := \sum_{j=1}^{\ell} r_j f_j$ denotes a random linear combination of f_1, \dots, f_ℓ (i.e., each r_j is chosen at random from \mathbb{F}), with high probability f also has distance at least δ from every polynomial of degree at most d_i over L_i . This statement is far from obvious, and to give a sense of why it is true, in Lemma 9.3 below we prove the following weaker statement that does *not* suffice to yield a tight analysis of FRI because it incurs a factor-of-2 loss in the distance parameter δ . Lemma 9.3 is due to Rothblum, Vadhan, and Wigderson [RVW13]; our proof follows the presentation of Ames et al. [AHIV17, Proof of Case 1 of Lemma 4.2] almost verbatim.

Lemma 9.3. *Let f_1, \dots, f_ℓ be a collection of ℓ functions on domain L_i , and suppose that at least one of the functions, say f_{j^*} , has relative distance at least δ from every polynomial of degree at most d_i over L_i . If $f := \sum_{j=1}^{\ell} r_j \cdot f_j$ denotes a random linear combination of f_1, \dots, f_ℓ , then with probability at least $1/|\mathbb{F}|$, f has distance at least $\delta/2$ from every polynomial of degree at most d_i over L_i .*

Proof. Let V denote the span of f_1, \dots, f_ℓ , i.e., V is the set of all functions obtained by taking arbitrary linear combinations of f_1, \dots, f_ℓ . Observe that a random element of V can be written as $\alpha \cdot f_{j^*} + x$ where α is a random field element and x is distributed independently of α . We argue that conditioned on any choice of x , there can be at most one choice of α such that $\alpha \cdot f_{j^*} + x$ has relative distance at most $\delta/2$ from some

polynomial of degree at most d_i . To see this, suppose by way of contradiction that $\alpha \cdot f_{j^*} + x$ has relative distance less than $\delta/2$ from some polynomial p of degree d_i and $\alpha' \cdot f_{j^*} + x$ has relative distance less than $\delta/2$ from some polynomial q of degree d_i , where $\alpha \neq \alpha'$. Then by the triangle inequality, $(\alpha - \alpha')f_{j^*}$ has relative distance less than $\delta/2 + \delta/2 = \delta$ from $p - q$. This contradicts the assumption that f_{j^*} has distance at least δ from every polynomial of degree at most d_i . \square

A line of work [RVW13, AHIV17, BKS18b, SGKS20, BCI⁺20] has improved Lemma 9.3 to avoid the factor-of-2 loss in the distance parameter δ . That is, rather than concluding that the random linear combination f has relative distance at most $\delta/2$ from every low-degree polynomial, these works show that f has relative distance at most δ from any low-degree polynomial. Two caveats are that these improvements do require δ to be “not too close to 1”, and they also have worse failure probability than the $1/|\mathbb{F}|$ probability appearing in Lemma 9.3—see Theorem 9.4 for details on these caveats.

Detailed Soundness Analysis Sketch for FRI. The soundness analysis overview provided here below is merely a sketch, and we direct the interested reader to [BKS18b, Section 7] for a very readable presentation of the full details. Suppose that the function G_0 over domain L_0 has relative distance more than δ from every degree d_0 polynomial. We must show that for every prover strategy, with high probability the prover fails at least one of the FRI verifier’s consistency checks during the Query Phase of FRI.

For any fixed value of x_0 chosen by the verifier, Equation (9.17) specifies a function G_1 over L_1 such that if the prover sends G_1 in round 1 of the Commitment Phase of FRI, then G_1 will always pass the verifier’s consistency check. Namely if for any $s_1 \in L_1$ we let $s_0, s'_0 \in L_0$ denote the two square roots of s_1 , then

$$G_1(s_1) = (x_0 - s_0) \cdot (s'_0 - s_0)^{-1} \cdot G_0(s'_0) + (x_0 - s'_0) \cdot (s_0 - s'_0)^{-1} \cdot G_0(s_0). \quad (9.18)$$

Note that G_1 depends on x_0 ; when we need to make this dependence explicit, we will write G_{1,x_0} rather than G_1 .

In round 1 of the Commitment Phase of FRI, a prover can hope to “luck out” in one of two ways. The first way is if the verifier happens to select a value $x_0 \in \mathbb{F}$ such that G_{1,x_0} has relative distance significantly less than δ from a polynomial of degree d_1 . The second way is that the prover could send a message $G'_1 \neq G_1$ such that G'_1 is much closer to a low-degree polynomial than is G_1 , and hope the verifier doesn’t “detect” the deviation from G_1 via its consistency checks.

It turns out that the second approach, of sending $G'_1 \neq G_1$, never increases the probability that the prover passes the verifier’s checks. Roughly speaking, this is because any “distance improvement” that the prover achieves by sending a function G'_1 that deviates from G_1 is compensated for by an increased probability that the prover fails the verifier’s consistency checks.

Let us now explain why the probability that the prover lucks out in the first sense is at most some small quantity ϵ_1 . The idea is that G_{1,x_0} is essentially a random linear combination of $G_{1,0}$ and $G_{1,1}$. Specifically, since G_{1,x_0} is a linear function in x_0 (see Equation (9.18)), we can write $G_{1,x_0} = G_{1,0} + x_0 \cdot G_{1,1}$. Since x_0 is chosen by the verifier uniformly at random from \mathbb{F} this means G_{1,x_0} is essentially a random linear combination of $G_{1,0}$ and $G_{1,1}$ (not quite, because the coefficient of $G_{1,0}$ is fixed to 1 rather than a random field element, but let us ignore this complication). Moreover, it is possible to show (though we omit the derivation) that if $G_{1,0}$ and $G_{1,1}$ are each of relative distance at most δ over L_1 from some polynomials $p(X)$ and $q(X)$ of degree less than $d_1 = d_0/2$, then G_0 is of relative distance at most δ over L_0 from the polynomial $p(X^2) + X \cdot q(X^2)$, which has degree less than $2d_1 = d_0$, contradicting our assumption. The strengthening of Lemma 9.3 discussed earlier in this section asserted that a random linear combination of two functions, at least one of which has relative distance at least δ from every polynomial of degree d_1 , is very likely to itself have relative distance at least δ from every such polynomial. Hence we reach the desired conclusion

that with high probability over the choice of x_0 , G_{1,x_0} has relative distance at least δ from every polynomial of degree at most d_1 .

The above analysis applies for every round i , not just to round $i = 1$. Specifically, the optimal prover strategy sends a specified function $G_{i,x_{i-1}}$ in each round i , and with high probability every $G_{i,x_{i-1}}$ has relative distance at least δ from a polynomial of degree at most d_i . If this holds for the final round i^* , then in each repetition of the Query Phase of FRI, the verifier's final consistency check will reject with probability at least $1 - \delta$. This is because $G_{i^*,x_{i^*-1}}$ will have relative distance δ from a constant function, and the prover in the final round of the commitment phase is forced to send a constant C with the verifier checking in each execution of the query phase whether $G_{i^*,x_{i^*-1}}(s_{i^*}) = C$ for a point s_{i^*} that is uniformly distributed in L_{i^*} .

In summary, conditioned on the optimal prover strategy not “lucking out” during the commitment phase (which happens with probability at most ε_1 by the analysis sketched above), each repetition of the query phase will reveal an inconsistency with probability at least $1 - \delta$. So the probability that all ℓ repetitions of the query phase fail to detect an inconsistency is $\varepsilon_2 = (1 - \delta)^\ell$. Hence, if G_0 has relative distance more than δ from any polynomial of degree at most d_0 , then the FRI verifier rejects with probability at least $1 - \varepsilon_1 - \varepsilon_2$. This is formalized in the following theorem from [BCI⁺20].

Theorem 9.4 ([BCI⁺20]). *Let $\rho = d_0/|L_0|$, $\eta \in (0, \sqrt{\rho}/20)$, and $\delta \in (0, 1 - \sqrt{\rho} - \eta)$. If FRI is applied to a function G_0 that has relative distance more than δ from any polynomial of degree at most d_0 , then the verifier accepts with probability at most $\varepsilon_1 + \varepsilon_2$, where $\varepsilon_2 = (1 - \delta)^\ell$ and $\varepsilon_1 = \frac{(d_0+1)^2}{(2\eta)^7 \cdot |\mathbb{F}|}$.*

To give a sense of how the parameters in Theorem 9.4 may be set, [BCI⁺20] work through a numerical example, setting ρ to $1/16$, $|\mathbb{F}|$ to be 2^{256} , η to 2^{-14} , and δ to $1 - \sqrt{\rho} - \eta = 3/4 - \eta$. They show that if d_0 is at most $2^{16} = 65536$, then with $\ell := 65$ invocations of the basic query phase, this yields to soundness error $\varepsilon_1 + \varepsilon_2 \leq 2^{-128}$. This is only sufficient to capture RICS instances with at most $d_0 = 65536$ nonzero entries; handling larger RICS instances at the same security level would require either a larger field, more repetitions, or lower rate.

Theorem 9.4 incentivizes protocol designers to set δ as large as possible, because larger values of δ lead to smaller values of ε_2 .⁹⁹ It is conjectured that an analog of Theorem 9.4 holds even for δ as large as roughly $1 - \rho$ (see for example [BBHR19, BCI⁺20]) rather than the $1 - \sqrt{\rho} - \eta$ bound on δ assumed in Theorem 9.4. Compared to the numerical example above, assuming these conjectures to be true allows a given soundness level (e.g., 2^{-128}) to be obtained with a significantly smaller repetition parameter ℓ and larger rate parameter ρ . This translates to an argument system with significantly smaller proofs and a faster prover. For this reason, several experimental evaluations of implemented argument systems based on FRI in fact assume the conjecture to be true [BBHR19, COS20].

9.5 From Reed-Solomon Testing to Multilinear Polynomial Commitments

Zhang et al. [ZXZS19] observe that, given any efficient IOP for Reed-Solomon proximity testing such as FRI, it is possible to devise a commitment scheme for multilinear polynomials in the IOP setting in the following manner. As observed in Lemma 3.8, evaluating an ℓ -variate multilinear polynomial q over \mathbb{F} at an input $r \in \mathbb{F}^\ell$ is equivalent to computing the inner product of the following two (2^ℓ) -dimensional vectors

⁹⁹Currently known analyses of implemented IOPs for RICS-satisfiability place restrictions on δ that are stronger than the requirement of Theorem 9.4 that $\delta < 1 - \sqrt{\rho} - \eta$. For example, [COS20, Theorem 8.2] requires that $\delta \leq (1 - \rho)/3 < 1/3$. Under such a restriction, FRI would require over 200 repetitions of the query phase to achieve soundness error less than 2^{-128} . Nonetheless, it is plausible that more careful analyses of the implemented IOPs are capable of removing these additional restrictions on δ .

$u_1, u_2 \in \mathbb{F}^{2^\ell}$. Associating $\{0, 1\}^\ell$ with $\{0, \dots, 2^\ell - 1\}$ in the natural way, the w 'th entry of the first vector u_1 is $q(w)$ and the w 'th entry of the second vector u_2 is $\chi_w(r)$, the w 'th Lagrange basis polynomial evaluated at r .

Let H be a multiplicative subgroup of \mathbb{F} of size $n := 2^\ell$. (A multiplicative subgroup of exactly this size only exists if and only if 2^ℓ divides $|\mathbb{F}| - 1$, so let us assume this). Let $b: H \rightarrow \{0, 1\}^\ell$ be a canonical bijection. To commit to a multilinear polynomial q , it suffices to commit to a *univariate* polynomial Q over \mathbb{F} of degree $|H| = n$ such that for all $a \in H$, $Q(a) = q(b(a))$, as q is fully specified by its evaluations over $\{0, 1\}^\ell$. To later reveal $q(z)$ at a point $z \in \mathbb{F}^\ell$ of the verifier's choosing, consider the vector u_2 containing all Lagrange basis polynomials evaluated at z . It suffices to confirm that

$$\sum_{a \in H} Q(a) \cdot u_2(a) = v, \tag{9.19}$$

where v is the claimed value of $q(z)$ and here we associate the $|H|$ -dimensional vector u_2 with a function over H in the natural way.

Let \hat{u}_2 be the unique polynomial of degree at most $|H|$ extending u_2 , and let

$$g(X) = Q(X) \cdot \hat{u}_2(X) - v \cdot |H|^{-1}.$$

Observe that Equation (9.19) holds if and only if $\sum_{a \in H} g(a) = 0$. Hence, Equation (9.19) can be checked by applying the univariate sum-check protocol described following Lemma 9.2 to the polynomial g . In more detail, in this protocol the prover sends a commitment to a polynomial h^* such that

$$g(X) = h^*(X) \cdot \mathbb{Z}_H(X) + X \cdot f(X)$$

for some polynomial f of degree at most $n - 1$, and the verifier applies FRI to check that the function

$$f(X) := (g(X) - h^*(X)\mathbb{Z}_H(X)) \cdot X^{-1}$$

has degree at most $n - 1$. Every time the FRI verifier needs to query $f(X)$ at an input r , the query can be answered so long as the verifier knows $Q(r)$, $\hat{u}_2(r)$, and $h^*(r)$.

The verifier can obtain the first and third values directly by querying the commitments to Q and h^* . However, evaluating $\hat{u}_2(r)$ requires time linear in n (Lemma 3.8). Fortunately, the computation $r \mapsto \hat{u}_2(r)$ is computed by a layered arithmetic circuit of size $O(n)$, depth $O(\log n)$, and a wiring pattern for which add_i and mult_i can be evaluated in $O(\log n)$ time for each layer i . Hence, the verifier can outsource the evaluation of $\hat{u}_2(r)$ to the prover using the GKR protocol (Section 4.6), with the verifier running in $O(\log^2 n)$ time.

9.6 An Alternative IOP-Based Polynomial Commitment: Ligerio

Recall from Section 9.2.2 that FRI can be built upon to give a polynomial commitment scheme, meaning it allows the prover to commit to a low-degree polynomial q , and the verifier can force the prover to later evaluate the committed polynomial q at any point $z \in \mathbb{F}$ of the verifier's choosing. Here is an alternative IOP achieving the same functionality. The scheme is implicit in work of Ames et al. [AHIV17], describing a system called Ligerio.

9.6.1 Identifying Tensor Product Structure in Polynomial Evaluation Queries

Let q be a degree- D univariate polynomial over field \mathbb{F}_p that the prover wishes to commit to, and let u denote the vector of coefficients of u . Then we can express evaluations of q as inner products of u with appropriate “evaluation vectors”. Specifically, if $q(X) = \sum_{i=0}^D u_i X^i$, then for $z \in \mathbb{F}_p$, $q(z) = \langle u, y \rangle$ where $y = (1, z, z^2, \dots, z^D)$ consists of powers of z , and $\langle u, y \rangle = \sum_{i=0}^D u_i v_i$ denotes the inner product of u and v . The vector y has a tensor-product structure in the following sense. Let us assume that $D + 1 = m^2$ is a perfect square, and define $a, b \in \mathbb{F}^m$ as $a := (1, z, z^2, \dots, z^{m-1})$ and $b := (1, z^m, z^{2m}, \dots, z^{m(m-1)})$. If we view y as an $m \times m$ matrix with entries indexed as $(y_{1,1}, \dots, y_{m,m})$, then y is simply the outer product $b \cdot a^T$ of a and b . That is, $y_{i,j} = z^{i \cdot m + j} = b_i \cdot a_j$ for all $0 \leq i, j \leq m - 1$.

The IOP-based polynomial commitment scheme implicit in [AHIV17] exploits the tensor-product structure within y . (Later, in Section 13.1.3, we will see another polynomial commitment scheme, based on hardness of the discrete logarithm problem, that also exploits this tensor structure). Here is how it works.

9.6.2 Description of the polynomial commitment scheme

Commitment Phase. Recalling that $[m]$ denotes the set $\{1, 2, \dots, m\}$, let us view the coefficient vector u of q as function mapping $[m] \times [m] \rightarrow \mathbb{F}$ in the natural way (much the way we viewed the vector y above as an $m \times m$ matrix). Let \hat{u} denote the unique bivariate extension polynomial of u of degree at most m in each variable. Let $\rho \in (0, 1)$ be a designated constant and L_0 be any designated subset of \mathbb{F} of size $\rho^{-1}m$. We denote the elements of L_0 by $\{\sigma_1, \dots, \sigma_{\rho^{-1}m}\}$.

The prover’s commitment to u will consist of a single message specifying a matrix M of size $m \times |L_0|$ with entry (i, j) equal to $\hat{u}(i, \sigma_j)$. That is, the i th row of M is simply a Reed-Solomon encoding of the i th row of the coefficient vector u when u is viewed as a matrix.

Upon receiving the commitment message, the verifier will interactively test it to confirm that each row of M is indeed (close to) a univariate polynomial of degree at most m . To do so, the verifier chooses a random vector $r \in \mathbb{F}^m$ and sends r to the prover. The prover responds with a vector $v \in \mathbb{F}^{|L_0|}$ claimed to equal $r^T \cdot M$. v is naturally interpreted as a function q_v over domain L_0 , with v_j interpreted as $q_v(\sigma_j)$.¹⁰⁰ The verifier reads v in its entirety, which costs at most $|L_0|$ queries.

For an integer parameter t that we will specify later, the verifier picks a set $Q \subseteq [|L_0|]$ of columns of M at random, and makes $t \cdot m$ queries to M to learn all entries of those columns of M . For each $i \in Q$, the verifier confirms that these columns are consistent with v , i.e., that $q_v(\sigma_i) = r^T \cdot M_i$ where M_i denotes the i th column of M .

Soundness Analysis for the Commitment Phase. If the message M passes the checks above, then the verifier can be confident that each row of M is close to a polynomial of degree at most m . This follows from Lemma 9.3, which states that when taking a random linear combination of functions, if even a single one of the functions is far from every degree d polynomial, then (with probability at least $1 - 1/|\mathbb{F}|$) so is the random linear combination. Quantitatively, if any row of M has relative distance more than δ from every polynomial of degree at most m , then Lemma 9.3 guarantees that, with probability at least $1 - 1/|\mathbb{F}|$, $r^T M$ has relative distance at least $\delta/2$ from every polynomial of degree at most m . In this event, since the verifier reads v in its entirety and confirms that v contains the evaluations of a polynomial q_v of degree at most m , $r^T M$ and v differ in at least a $\delta/2$ fraction of their entries. Hence, the probability that $v_i = r^T \cdot M_i$ for all

¹⁰⁰The prover could instead send the $m + 1$ coefficients of q_v rather than all $|L_0|$ evaluations of q_v over the set L_0 . This would improve the proof length and query cost by a factor of ρ^{-1} . As ρ is a constant in $(0, 1)$, this does not affect the asymptotic costs of the IOP.

$i \in Q$ is at most $(1 - \delta/2)^t$. We will set t to ensure that this probability is below some desired soundness level, e.g., to ensure that $(1 - \delta/2)^t \leq 2^{-128}$. In summary, we have established the following.

Claim 9.5. *If the verifier's checks in the Commitment Phase all pass with probability more than $1/|\mathbb{F}| + (1 - \delta/2)^t$, then each row i of M has relative distance at most δ from some polynomial p_i of degree at most m .*

Let us assume henceforth that $\delta < (1 - \rho)/2$. This assumption combined with Claim 9.5 ensures that if the prover passes the verifier's checks with probability more than $(1 - \delta/2)^t$, then for each row M_i of M , there is a *unique* polynomial p_i of degree at most m at relative distance at most δ from the i th row of M .

Actually, we will need a refinement of Claim 9.5. Claim 9.5 asserts that the verifier can be confident that each row i of M has relative distance at most δ from p_i . Let $E_i = \{j: p_i(\sigma_j) \neq M_{i,j}\}$ denote the subset of L_0 on which p_i and row i of M differ, and let $E = \cup_{i=1}^m E_i$. That is, E is the set of columns such that at least one row i deviates from its closest polynomial p_i in that column.

Claim 9.5 doesn't rule out the possibility $|E| = \sum_{i=1}^m |E_i|$. In other words, it leaves open the possibility that any two different rows i, i' of M deviate from the corresponding low-degree polynomials $p_i, p_{i'}$ in different locations. We need a refinement of Claim 9.5 that does rule out this possibility. We do not prove this refinement—the interested reader can find the proof in [AHIV17, Lemma 4.2].

Claim 9.6. (Ames et al. [AHIV17, Lemma 4.2]) *Suppose that $\delta < \frac{1-\rho}{4}$ and that the verifier's checks in the Commitment Phase all pass with probability more than $\varepsilon_1 := |L_0|/|\mathbb{F}| + (1 - \delta)^t$. Let $E = \cup_{i=1}^m E_i$ be defined as above. Then $|E| \leq \delta \cdot |L_0|$.*

Evaluation Phase. Suppose the verifier requests that the prover reveal $q(z) = \langle u, y \rangle$ where u and y are defined as in Section 9.6.1. Note that viewing u and y as matrices, $q(z) = a^T \cdot u \cdot b$, where $a, b \in \mathbb{F}^m$ are also as defined in Section 9.6.1. The evaluation phase is entirely analogous to the commitment phase, except that the random vector r used in the commitment phase is replaced with a .

In more detail, the prover first sends the verifier a polynomial $h(X)$ of degree at most m claimed to equal $\sum_{i=1}^m a_i \cdot p_i(X)$, where as in the soundness analysis of the Commitment Phase, p_i denotes the closest polynomial of degree at most m to row i of M (if the prover is honest, then $p_i(\sigma_j) = M_{i,j}$ for all $(i, j) \in [m] \times [L_0]$). The verifier picks a set $Q' \subseteq [L_0]$ of size t and checks that for all $j \in Q'$, $h(\sigma_j) = \sum_{i=1}^m a_i \cdot M_{i,j}$. This costs $t \cdot m$ queries to M . If the verifier's checks all pass, then the verifier outputs $\sum_{j=1}^m b_j \cdot h(j)$ as the evaluation $q(z)$ of the committed polynomial.

Completeness and Binding of the Commitment Scheme. Clearly if the prover sends the matrix M that actually contains the coefficient vector of q , and then sends the prescribed messages throughout the remainder of the Commitment and Evaluation phases, the verifier's checks will pass and the verifier will output $q(z)$.

To argue binding, let $h'(X) := \sum_{i=1}^m a_i \cdot p_i(X)$. We claim that, if the prover passes the verifier's checks in the Commitment Phase with probability more than $\varepsilon_1 = |L_0|/|\mathbb{F}| + (1 - \delta)^t$ and sends a polynomial $h(X)$ in the Evaluation Phase such that $h(X) \neq h'(X)$, then the prover will pass the verifier's checks in the Evaluation Phase with probability at most $\varepsilon_2 := (\delta + \rho)^t$. To see this, observe that $h(X)$ and $h'(X)$ are two distinct polynomials of degree at most m , and hence they can agree on at most m inputs. Denote this agreement set by A . The verifier rejects in the Evaluation Phase if there is any $j \in Q'$ such that $j \notin A \cup E$, where E is as in Claim 9.6. $|A \cup E| \leq |A| + |E| \leq m + \delta \cdot |L_0|$, and hence a randomly chosen column j of M is in $A \cup E$ with probability at most $m/|L_0| + \delta \leq \rho + \delta$. It follows that the verifier will reject with probability at least $1 - (\rho + \delta)^t$.

In summary, we have shown that if the prover passes the verifier's checks in the Commitment Phase with probability at least $|L_0|/|\mathbb{F}| + (1 - \delta)^t$, then the prover is *bound* to the polynomial $q^*(z) = \sum_{0 \leq i, j \leq m-1} c_{i,j} z^{i+m+j}$,

where $c_{i,j} = p_i(\sigma_j)$, in the sense that the verifier either outputs $q^*(z)$ on Evaluation query z , or rejects in the Evaluation Phase with probability at least $1 - (\rho + \delta)^t$.

9.6.3 Discussion of Costs

Let λ denote a security parameter defined as follows. Suppose we wish to guarantee that if the prover convinces the verifier not to reject in either the Commitment Phase or Query Phase with probability at least $\epsilon_1 + \epsilon_2 = 2^{-\lambda}$, then the prover is forced to answer any Evaluation query consistent with a fixed polynomial q^* of degree at most D . The costs of the polynomial commitment scheme are then as follows. Throughout, we suppress dependence on ρ^{-1} and δ , as we consider these parameters to be constants in $(0, 1)$.

Message Lengths and Query Complexity. The Commitment Phase consists of three messages. The first message, from the prover to verifier, has length $O(D)$. The second message, from verifier to prover, has length $O(\sqrt{D})$. The third message, from prover to verifier, has length $O(\lambda \cdot \sqrt{D})$. The verifier makes $O(\lambda \sqrt{D})$ queries to the prover's first message, and reads the prover's final message in its entirety.

The Evaluation Phase consists of two messages. The first, from verifier to prover, simply specifies the evaluation point z . The second, from prover to verifier, has length $O(\sqrt{D})$. In this phase, the verifier reads the prover's message in its entirety, and makes $O(\lambda \sqrt{D})$ additional queries to the prover's first message from the Commitment Phase.

When transformed into an argument system using Merkle-hashing, this IOP leads to proofs consisting of $O(\lambda \sqrt{D})$ field elements and $O(\sqrt{D})$ Merkle-authentication paths. These authentication paths consist of $O(\sqrt{D} \log D)$ cryptographic hash evaluations.

Prover Time. The prover's runtime in the argument resulting from this IOP is dominated by two operations. The first is to compute the Reed-Solomon encoding of each row of the matrix M . This requires one FFT operation per row, on vectors of length $\Theta(\sqrt{D})$. Since each such FFT requires $O(\sqrt{D} \log D)$ field operations, the total runtime for the FFTs is $O(D \log D)$ field operations. The second bottleneck is the need for the prover to compute a Merkle-hash of its first message, which has length $O(D)$. This requires $O(D)$ cryptographic hash evaluations.

Verifier Time. The verifier's runtime in the argument system is dominated by the need to check that the $O(\sqrt{D} \log D)$ cryptographic hash evaluations in the proof are correctly executed, and the $O(\lambda \sqrt{D})$ field operations required to simulate the IOP verifier's checks on the prover's messages.

Comparison to FRI. The communication complexity and verifier runtime of the argument system is much larger than that of FRI, at least asymptotically. Whereas the FRI verifier performs $O(\lambda \log D)$ field operations and $O(\log^2 D)$ cryptographic hash evaluations, the verifier of this section performs $O(\lambda \sqrt{D})$ field operations and $O(\sqrt{D} \log D)$ cryptographic hash evaluations, with similar differences in proof length.

The prover runtime in FRI and the protocol of this section are asymptotically similar, in that both are dominated by $O(D \log D)$ field operations coming from FFTs and $O(D)$ hash operations to build a Merkle tree over a vector of length $O(D)$. The difference is that here the prover is doing many, smaller FFTs whereas in FRI the prover is doing a single FFT on a vector of length D . While these result in the same asymptotic runtimes, FFTs are notoriously space intensive and difficult to distribute. Hence, performing $O(\sqrt{D})$ FFTs on vectors of length $O(\sqrt{D})$ may lead to better prover scalability than performing a single FFT on a vector of length $O(D)$.

A multilinear polynomial commitment. The polynomial commitment of this section applies also to multilinear rather than univariate polynomials. This is because evaluation of a multilinear polynomial q can be formulated as an inner product of a coefficient vector u with an evaluation vector y possessing the requisite tensor-product structure described in Section 9.6.1. See Section 13.1 for details on this formulation. In contrast, FRI is only known to yield a multilinear polynomial commitment in a more indirect manner (Section 9.5).

Chapter 10

Zero-Knowledge Proofs and Arguments

10.1 What is Zero-Knowledge?

The definition of a zero-knowledge proof or argument captures the notion that the verifier should learn nothing from the prover other than the validity of the statement being proven.¹⁰¹ That is, any information the verifier learns by interacting with the honest prover could be learned by the verifier on its own without access to a prover. This is formalized via a simulation requirement, which demands that there be an efficient algorithm called the *simulator* that, given only as input the statement to be proved, produces a distribution over transcripts that is indistinguishable from the distribution over transcripts produced when the verifier interacts with an honest prover (recall from Section 3.1 that a transcript of an interactive protocol is a list of all messages exchanged by the prover and verifier during the execution of the protocol).

Definition 10.1 (Informal definition of zero-knowledge). A proof or argument system with prescribed prover \mathcal{P} and prescribed verifier \mathcal{V} for a language \mathcal{L} is said to be zero-knowledge if for any probabilistic polynomial time verifier strategy \hat{V} , there exists a probabilistic polynomial time algorithm S (which can depend on \hat{V}), called the simulator, such that for all $x \in \mathcal{L}$, the distribution of the output $S(x)$ of the simulator is “indistinguishable” from $\text{View}_{\hat{V}}(\mathcal{P}(x), \hat{V}(x))$. Here, $\text{View}_{\hat{V}}(\mathcal{P}(x), \hat{V}(x))$ denotes the distribution over transcripts generated by the interaction of prover strategy \mathcal{P} and verifier strategy \hat{V} within the proof or argument system.

Informally, the existence of the simulator means that, besides learning that $x \in \mathcal{L}$, the verifier \mathcal{V} does not learn anything from the prover beyond what \mathcal{V} could have efficiently computed herself. This is because, conditioned on x being in \mathcal{L} , \mathcal{V} cannot tell the difference between generating a transcript by interacting with the honest prover, vs. generating the transcript by ignoring the prover and instead running the simulator. Accordingly, any information the verifier could have learned from the prover could also have been learned from the simulator (which is an efficient procedure, and hence the verifier can afford to run the simulator herself).

In Definition 10.1, there are three natural meanings of the term “indistinguishable”.

- One possibility is to require that $S(x)$ and $\text{View}_{\hat{V}}(\mathcal{P}(x), \hat{V}(x))$ are literally the same distribution. In this case, the proof or argument system is said to be *perfect-zero knowledge*.¹⁰²

¹⁰¹Recall that a *proof* satisfies statistical soundness, while an argument satisfies computational soundness. See Definitions 3.1 and 3.2.

¹⁰²In the context of perfect zero-knowledge *proofs*, it is standard to allow the simulator to abort with probability up to $1/2$, and

- Another possibility is to require that the distributions $S(x)$ and $\text{View}_{\hat{V}}(\mathcal{P}(x), \hat{V}(x))$ have negligible *statistical distance*. In this case, the proof or argument system is said to be *statistical zero-knowledge*. Here, the statistical distance (also called total variation distance) between two distributions D_1 and D_2 is defined to be

$$\frac{1}{2} \sum_y |\Pr[D_1(x) = y] - \Pr[D_2(x) = y]|,$$

and it equals the maximum over all algorithms \mathcal{A} (including inefficient algorithms) of

$$\left| \Pr_{y \leftarrow D_1} [\mathcal{A}(y) = 1] - \Pr_{y \leftarrow D_2} [\mathcal{A}(y) = 1] \right|,$$

where $y \leftarrow D_i$ means that y is a random draw from the distribution D_i . Hence, if two distributions have negligible statistical distance, then no algorithm (regardless of its runtime) can distinguish the two distributions with non-negligible probability given a polynomial number of samples from the distributions.

- The third possibility is to require that all *polynomial time* algorithms \mathcal{A} cannot distinguish the distributions $S(x)$ and $\text{View}_{\hat{V}}(\mathcal{P}(x), \hat{V}(x))$ except with negligible probability, when given as input a polynomial number of samples from the distributions. In this case, the proof or argument system is said to be *computational zero-knowledge*.

Accordingly, when someone refers to a “zero-knowledge protocol”, there are actually *at least* 6 types of protocols they may be referring to. This is because soundness comes in two flavors—statistical (proofs) and computational (arguments)—and zero-knowledge comes in at least 3 flavors (perfect, statistical, and computational). In fact, there are even more subtleties to be aware of when considering how to define the notion of zero-knowledge.

- **(Honest vs. dishonest verifier zero-knowledge).** Definition 10.1 requires an efficient simulator for *every possible* probabilistic polynomial time verifier strategy \hat{V} . This is referred to as malicious- or dishonest-verifier- zero knowledge (though papers often omit the clarifying phrase malicious or dishonest-verifier). It is also interesting to consider only requiring an efficient simulator for the prescribed verifier strategy \mathcal{V} . This is referred to as *honest-verifier zero-knowledge*.
- **(Plain zero-knowledge vs. auxiliary-input zero-knowledge).** Definition 10.1 considers the verifier \hat{V} to have only one input, namely the public input x . This is referred to as *plain zero-knowledge*, and was the original definition given in the conference paper of Goldwasser, Micali, and Rackoff [GMR89] that introduced the notion of zero-knowledge (along with the notion of interactive proofs). However, when zero-knowledge proofs and arguments are used as subroutines within larger cryptographic protocols, one is typically concerned about dishonest verifiers that may compute their messages to the prover based on information acquired from the larger protocol prior to executing the zero-knowledge protocol. To capture such a setting, one must modify Definition 10.1 to refer to verifier strategies \hat{V} that take two inputs: the public input x known to both the prover and verifier, and an *auxiliary input* z known only to the verifier and simulator, and insist that the output $S(x, z)$ of the simulator is “indistinguishable” from $\text{View}_{\hat{V}}(\mathcal{P}(x), \hat{V}(x, z))$. This modified definition is referred to as *auxiliary-input*

require $S(x)$ to be distributed identically to $\text{View}_{\hat{V}}(\mathcal{P}(x), \hat{V})$ conditioned on $S(x)$ not aborting. This is because, if the simulator is not permitted to abort, no perfect-zero knowledge proofs are known for any non-trivial problems (meaning problems not known to be in **BPP**, the class of problems solvable in randomized polynomial time). This subtlety will not be relevant to this survey.

zero-knowledge. Of course, the distinction between auxiliary-input and plain zero-knowledge is only relevant when considering dishonest verifiers.

An added benefit of considering auxiliary-input computational zero-knowledge is that this notion is closed under sequential composition. This means that if one runs several protocols satisfying auxiliary-input computational zero-knowledge, one after the other, the resulting protocol remains auxiliary-input computational zero-knowledge. This is actually not true for plain computational zero-knowledge, though known counterexamples are somewhat contrived. The interested reader is directed to [BV10] and the references therein for a relatively recent study of the composition properties of zero-knowledge proofs and arguments.

The reader may be momentarily panicked at the fact that we have now roughly 24 notions of zero-knowledge protocols, one for every possible combination of (statistical vs. computational soundness), (perfect vs. statistical vs. computational zero-knowledge), (honest-verifier vs. dishonest-verifier zero-knowledge), and (plain vs. auxiliary input zero-knowledge). That's $2 \cdot 3 \cdot 2 \cdot 2$ combinations in total, though for honest-verifier notions of zero-knowledge the difference between auxiliary-input and plain zero-knowledge is irrelevant. Fortunately for us, there are only a handful of variants that we will have reason to study in this manuscript, summarized below.

In Sections 10.2-10.4 below, we briefly discuss statistical zero-knowledge proofs. Our discussion is short because, as we explain, of statistical zero-knowledge proofs are not very powerful (e.g., while they are capable of solving some problems believed to be outside of **BPP**, they are not believed to be able to solve **NP**-complete problems). Roughly all we do is describe what is known about their limitations, and then give a sense of what they are capable of computing by presenting two simple examples: a classic zero-knowledge proof system for graph non-isomorphism due to [GMW91] (Section 10.3), and a particularly elegant protocol for a problem called the Collision Problem (this problem is somewhat contrived, but the protocol is an instructive example of the power of zero-knowledge).

In subsequent chapters, we present a variety of perfect zero-knowledge arguments. All are non-interactive (possibly after applying the Fiat-Shamir transformation), rendering the distinction between malicious- and honest-verifier (and auxiliary-input vs. plain) zero-knowledge irrelevant.^{103,104}

Remarks on simulation. A common source of confusion for those first encountering zero-knowledge is to wonder whether an efficient simulator for the honest verifier's view in a zero-knowledge proof or argument for a language \mathcal{L} implies that the problem can be solved by an efficient algorithm (with no prover). That is, given input x , why can't one run the simulator S on x several times and try to discern from the transcripts output by S whether or not $x \in \mathcal{L}$? The answer is that this would require that for every pair of inputs (x, x') with $x \in \mathcal{L}$ and $x' \notin \mathcal{L}$, the distributions $S(x)$ and $S(x')$ are efficiently distinguishable. Nothing in the

¹⁰³More precisely, when the Fiat-Shamir transformation is applied to an honest-verifier zero-knowledge proof or argument and is instantiated in the plain model (by replacing the random oracle with a concrete hash function), the resulting non-interactive argument is zero-knowledge so long as the hash family used to instantiate the random oracle satisfies a property called programmability. This applies even to dishonest verifiers, since non-interactive protocols leave no room for misbehavior on the part of the verifier. When working in the random oracle model instead of the plain model, there are some subtleties regarding how to formalize zero-knowledge that we elide in this survey (the interested reader can find a discussion of these subtleties in [Pas03, Wee09]).

¹⁰⁴For non-interactive arguments that use a structured reference string (SRS), such as the one we describe later in Section 14.5.5, one may consider (as an analog of malicious-verifier zero-knowledge) settings in which the SRS is not generated properly. For example, the notion of *subversion zero knowledge* demands that zero-knowledge be maintained even when the SRS is chosen maliciously. SNARKs that we describe in this survey that use an SRS can be tweaked to satisfy subversion zero-knowledge [BFS16, ABLZ17, Fuc18]. On the other hand, it is not possible for a SNARK for circuit satisfiability to be sound in the presence of a maliciously chosen SRS if the SNARK is zero-knowledge [BFS16].

definition of zero-knowledge guarantees this (in fact, the definition of zero-knowledge says nothing about how that simulator S behaves on inputs x' that are not in \mathcal{L}).

Indeed, it is entirely possible that an efficient simulator S can produce accepting transcripts for a zero-knowledge protocol even when run on inputs $x' \notin \mathcal{L}$ (similarly, in the context of zero-knowledge proofs *of knowledge*, where the prover is claiming to *know* a witness w satisfying some property, the simulator will be able to produce accepting transcripts *without* knowing a witness).

One reason this can hold is that a zero-knowledge protocol may be interactive, yet the simulator only needs to produce convincing *transcripts* of the interaction. This means that the simulator is able to do things like first choose all of the verifier's challenges, and then choose all of the prover's messages in a manner that depends on those challenges. In contrast, a cheating prover must send its message in each round prior to learning the verifier's challenge in that round. This will be the situation for the simulators we construct in Section 10.4 for the Collision Problem, and the zero-knowledge proofs of knowledge that we develop in Section 11.2 (e.g., in Schnorr's protocol for establishing knowledge of a discrete logarithm).

A second possible reason that a simulator may not allow a prover to convince the verifier to accept false claims is that, if the zero-knowledge proof is not public coin, then the simulator can choose the verifier's private coins and use its knowledge of the private coins to produce accepting transcripts, while a cheating prover does not have access to the verifier's private coins. This will be the case for the simulator for the graph non-isomorphism protocol given in Section 10.3.

For the above reasons, the existence of an efficient simulator for a protocol is no barrier to soundness or knowledge-soundness of the protocol.

10.2 The Limits of Statistical Zero Knowledge Proofs

It is known that any language solvable by a statistical zero-knowledge proof with a polynomial time verifier is in the complexity class $\mathbf{AM} \cap \mathbf{coAM}$ [AH91, For87]. This means that such proof systems are certainly no more powerful than *constant-round* (non-zero-knowledge) interactive proofs, and are unlikely to be able to solve \mathbf{NP} -complete problems.¹⁰⁵ In contrast, the SNARKs we give in this survey with polynomial time verifiers are capable of solving problems in \mathbf{NEXP} , a vastly bigger class than \mathbf{NP} (and with linear-time verifiers and logarithmic proof length, the SNARKs in this survey can solve \mathbf{NP} -complete problems). The upshot is that statistical zero-knowledge proof systems are simply not powerful enough to yield efficient general-purpose protocols (i.e., to verifiably outsource arbitrary witness-checking procedures in zero-knowledge). Accordingly, we will discuss statistical zero-knowledge proofs only briefly in this survey. The reason we discuss them at all is because they do convey some intuition about the power of zero-knowledge that is useful even once we turn to the more powerful setting of (perfect honest-verifier) zero-knowledge *arguments*.

10.3 Honest-Verifier SZK Protocol for Graph Non-Isomorphism

Two graphs G_1, G_2 on n vertices are said to be isomorphic if they are the same graph up to labelling of vertices. Formally, for a permutation $\pi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, let $\pi(G_i)$ denote the graph obtained by replacing each edge (u, v) with $(\pi(u), \pi(v))$. Then G_1 is isomorphic to G_2 if there exists a permutation π such that $\pi(G_1) = G_2$. That is, π is an isomorphism between G_1 and G_2 so long as $(i, j) \in G_1$ if and only if $(\pi(i), \pi(j))$ is an edge of G_2 .

¹⁰⁵If $\mathbf{AM} \cap \mathbf{coAM}$ contains \mathbf{NP} -complete problems, then $\mathbf{AM} = \mathbf{coAM}$, which many people believe to be false. That is, the existence of efficient one-round proofs of membership in a language does not seem like it should necessarily imply the existence of efficient one-round proofs of *non*-membership in the same language.

There is no known polynomial time algorithm for the problem of determining whether two graphs are isomorphic (though a celebrated recent result of Babai [Bab16] has given a *quasipolynomial* time algorithm for the problem). In Protocol 2, we give a perfect honest-verifier zero-knowledge protocol for demonstrating that two graphs are *not* isomorphic, due to seminal work of Goldreich, Micali, and Wigderson [GMW91]. Note that even obtaining a *nonzero-knowledge* protocol is not obvious for this problem. While a (nonzero-knowledge) proof that two graphs *are* isomorphic can simply specify the isomorphism π ,¹⁰⁶ it is not clear that there is a similar witness for the *non-existence* of any isomorphism.

Protocol 2 Honest-verifier perfect zero-knowledge protocol for graph non-isomorphism

Verifier picks $b \in \{1, 2\}$ at random, and chooses a random permutation $\pi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$.
 Verifier sends $\pi(G_b)$ to prover.
 Prover responds with b' .
 Verifier accepts if $b' = b$ and rejects otherwise.

We now explain that the protocol in Protocol 2 is perfectly complete, has soundness error at most $1/2$, and is honest-verifier perfect zero-knowledge.

Perfect Completeness. If G_1 and G_2 are not isomorphic, then $\pi(G_i)$ is isomorphic to G_i but not to G_{3-i} . Hence, the prover can identify b from $\pi(G_b)$ by determining which of G_1, G_2 it is that $\pi(G)$ is isomorphic to.

Soundness. If G_1 and G_2 are isomorphic, then $\pi(G_1)$ and $\pi(G_2)$ are identically distributed when π is a permutation chosen uniformly at random over the set of all $n!$ permutations over $\{1, \dots, n\}$. Hence, statistically speaking, the graph $\pi(G_b)$ provides no information as to the value of b , which means regardless of the prover's strategy for selecting b' , b' will equal b with probability exactly $1/2$. The soundness error can be reduced to 2^{-k} by repeating the protocol k times sequentially.

Perfect honest-verifier zero-knowledge. Intuitively, when the graphs are not isomorphic, the honest verifier cannot possibly learn anything from the prover because the prover just sends the verifier a bit b' equal to the bit b that the verifier selected on its own. Formally, consider the simulator that on input (G_1, G_2) , simply chooses b at random from $\{1, 2\}$ and chooses a random permutation π , and outputs the transcript $(\pi(G_b), b)$. This transcript is distributed identically to the honest verifier's view when interacting with the prescribed prover.

Discussion of zero-knowledge. We remark that the protocol is not zero-knowledge against malicious verifiers (assuming there is no polynomial time algorithm for graph isomorphism¹⁰⁷). Imagine a dishonest verifier that somehow knows a graph H that is isomorphic to one of G_1, G_2 , but the verifier does not know which. If the verifier replaces its prescribed message $\pi(G_b)$ in the protocol with the graph H , then the honest prover will reply with the value b' such that H is isomorphic to $G_{b'}$. Hence, this dishonest verifier learns which of the two input graphs H is isomorphic to, and if there is no efficient algorithm for graph isomorphism, then this is information that the verifier could not have computed efficiently on its own.

It is possible to transform this protocol into a proof that is zero-knowledge even against dishonest verifiers. Our sketch of this result follows [Gol07, Section 4.7.4.3]. The rough idea is that if the verifier only sends query graphs H to the prover such that the verifier *already knows* which of G_1, G_2 it is that H is isomorphic to, then the verifier cannot possibly learn anything from the prover's response (as the prover's

¹⁰⁶A perfect zero-knowledge proof for graph isomorphism is also known, an exposition of which can be found in [Gol07, Section 4.3.2].

¹⁰⁷Though it may not be terribly surprising if this assumption turns out to be false, in light of the recent result of Babai [Bab16].

response is simply a bit b' such that H is isomorphic to $G_{b'}$). Hence, we can insist that the verifier first *prove to the prover* that the verifier knows a bit b such that H is isomorphic to G_b .

For this approach to preserve soundness, it is essential that the verifier's proof not leak any information to the prover about b (i.e., the verifier's proof to the prover should itself be zero-knowledge, or at least satisfy a weaker property called witness-independence (see [Gol07, Section 4.6])). This is because, if G_1 and G_2 are isomorphic (i.e., the prover is lying when it claims that G_1 and G_2 are not isomorphic), a cheating prover could use information leaked from the verifier's proof about bit b in order to guess the value of b with probability more than $1/2$.

Of course, we are omitting many details of how the verifier might prove to the prover in zero-knowledge that it knows a b such that H is isomorphic to G_b . But hopefully this gives some sense of how one might transform honest-verifier zero-knowledge proofs into dishonest-verifier proofs. Clearly, the resulting dishonest-verifier zero-knowledge protocol is more expensive than the honest-verifier zero-knowledge one, because achieving zero-knowledge against dishonest verifiers requires the execution of a second zero-knowledge proof (with the role of prover and verifier reversed).

10.4 Honest-Verifier SZK Protocol for the Collision Problem

In the Collision Problem, the input is a list (x_1, \dots, x_N) of N numbers from a range of size $R = N$ (while the list length and the range size are equal, it is helpful to distinguish the two quantities, with N referring to the former and R referring to the latter). The goal of the problem is to determine whether every range element appears in the list. Since $R = N$, this holds if and only if every range element appears *exactly* once in the list. However, there is a twist to make the problem easier: it is assumed that *either* every range element appears exactly once in the list (call such inputs YES instances), *or* exactly $R/2$ range elements appear twice in the list (this means, of course, that the other $R/2$ range elements do not appear in the list at all). Call such inputs NO instances. Algorithms for the Collision Problem are allowed to behave arbitrarily on inputs that fail to satisfy the above assumption.

The name Collision Problem refers to the fact that if the input list is interpreted as the evaluation table of a function h mapping domain $\{1, \dots, N\}$ to range $\{1, \dots, R\}$, then YES instances have no collisions (i.e., $h(i) \neq h(j)$ unless $i = j$), while NO instances have many collisions (there are $N/2$ pairs (i, j) such that $h(i) = h(j)$ yet $i \neq j$). This problem was originally introduced as a loose/idealized model of the task of finding collisions in a cryptographic hash function h .¹⁰⁸ With this interpretation as motivation, for each range element $k \in \{1, \dots, R\}$, we refer to any i with $x_i = k$ as a *pre-image* of k .

In the Collision Problem, since N is thought of as modeling the domain size and range size of a cryptographic hash function, we consider N to be “exponentially large”.¹⁰⁹ Accordingly, for this problem, an algorithm should be considered “efficient” (i.e., “polynomial time”) only if it runs in time $\text{polylog}(N)$.

Fastest Algorithm with no Prover. It is known that the fastest possible algorithm for the Collision problem runs in time $\Theta(\sqrt{N})$ (see Footnote 78 in Section 6.3.2.2), i.e., there is no “efficient” algorithm for the Collision Problem. We briefly sketch how to show this. The best algorithm simply inspects $c \cdot \sqrt{N}$ randomly

¹⁰⁸A key difference between finding collisions in a real-world cryptographic hash function h and the Collision Problem is that in the former task, h will have a succinct implicit description (e.g., via a computer program that on input i quickly outputs $h(i)$), while in the Collision Problem h does not necessarily have a description that is shorter than the list of all of its evaluations.

¹⁰⁹Strictly speaking, this is a misnomer, because the size of the input to the Collision Problem is N . But the Collision Problem is modeling a setting where the size of the input (namely, the description of a cryptographic hash function h with domain size and range size N) is really $\text{polylog}(N)$.

chosen list elements for a sufficiently large constant $c > 0$, and outputs 1 if they are all distinct, and outputs 0 otherwise. Clearly, when run on a YES instance, the algorithm outputs 1 with probability 1, since for YES instances every list element is distinct. Whereas when run on a NO instance, the birthday paradox implies that for a large enough constant $c > 0$, there will be a “collision” in the sampled list elements with probability at least $1/2$.¹¹⁰ This runtime is optimal up to a constant factor, because it is known that any algorithm that “inspects” $\ll \sqrt{N}$ list elements cannot effectively distinguish YES instances from NO instances (intuitively, this is because any algorithm that inspects fewer than $\Theta(\sqrt{N})$ list elements of a random NO instance will with probability $1 - o(1)$ fail to find a collision, and in this case the algorithm has no way to tell the input apart from a random YES instance).¹¹¹

HVSZK Protocol with Efficient Verifier. Here is an honest-verifier statistical zero-knowledge proof for the Collision Problem. The protocol consists of just one round (one message from verifier to prover and one reply from prover to verifier), and the verifier runs in time just $O(\log N)$ (both messages consist of $\log N$ bits, and to check the proof the verifier inspects only *one* element of the input list).

The first message of the protocol, from verifier to prover, consists of a random range element $k \in \{1, \dots, R\}$. The prover responds with a pre-image i of k . The verifier simply checks that indeed $x_i = k$, outputting ACCEPT if so and REJECT otherwise.

We now explain that the protocol is complete, sound, and honest-verifier perfect zero-knowledge (this means there is a simulator running in time $\text{polylog}(N)$ that, on any YES instance, produces a distribution over transcripts identical to that of that of the honest verifier interacting with the honest prover). Completeness is clear because for YES instances, each range element appears once in the input list, and hence regardless of which range element $k \in \{1, \dots, R\}$ is selected by the verifier, the prover can provide a pre-image of k . Soundness holds because for NO instances, $R/2$ range elements do not appear at all in the input list, and hence with probability $1/2$ over the random choice of $k \in \{1, \dots, R\}$, it will be impossible for the prover to provide a pre-image of k .

To establish honest-verifier perfect zero-knowledge, for any YES instance (x_1, \dots, x_N) , we have to give an efficient simulator that generates transcripts distributed identically to those generated by the honest verifier interacting with the honest prover. The simulator picks a random domain item $i \in \{1, \dots, N\}$, and outputs the transcript (x_i, i) . Clearly, the simulator runs in logarithmic time (it simply chooses i , which consists of $\log N$ bits, and inspects one element of the input list, namely x_i). Since in any YES instance, each range element appears exactly once in the input list, picking a random domain item $i \in \{1, \dots, N\}$ and outputting the transcript (x_i, i) yields the same distribution over transcripts as picking a random range element k and outputting (x_i, i) where i is the unique pre-image of k . Hence, on YES instances, the simulator’s output is distributed identically to the view of the honest verifier interacting with the honest prover. Put more intuitively, the honest verifier in this protocol, when run on a YES instance, simply learns a random pair (x_i, i) where i is chosen at random from $\{1, \dots, N\}$, and this is clearly information the verifier could have efficiently computed on its own, by choosing i at random and inspecting x_i .

Discussion. This protocol is included in this survey because it cleanly elucidates some of the counter-intuitive features of zero-knowledge protocols.

¹¹⁰Let $c = 2$. If there is a collision within the first \sqrt{N} samples, we are done. Otherwise, the probability that none of the first \sqrt{N} sampled range elements appear within the second \sqrt{N} sampled range elements is at most $(1 - \sqrt{N}/N)^{\sqrt{N}} = (1 - 1/\sqrt{N})^{\sqrt{N}} \approx 1/e < 1/2$.

¹¹¹The expected number of collisions observed on a random NO instance after inspecting at most T items of the input list is $O(T^2/N)$, so if $T \leq o(\sqrt{N})$ this expectation is $o(1)$. Markov’s inequality then implies that with probability $1 - o(1)$, no collision is observed by the algorithm.

- The simulator, even if run on a NO instance, will always output an accepting transcript (x_i, i) . This fact may initially feel like it contradicts soundness of the protocol. However, it does not. This is because, if run on a NO instance, the simulator picks the verifier challenge x_i specifically to be an “answerable” challenge, i.e., a range element that appears in the input list. Whereas the actual verifier would have chosen a *random* range element as a challenge, which on a NO instance will, with probability $1/2$, have no pre-image and hence not be answerable.
- The existence of an efficient simulator is no barrier to intractability of the problem. While the simulator runs in time $O(\log N)$, the fastest algorithm for the problem requires time $\Theta(\sqrt{N})$.
- While the protocol is honest-verifier zero-knowledge, it is not dishonest-verifier zero-knowledge. Indeed, a dishonest verifier can “use” the honest prover to solve the problem of finding a pre-image of a specific range element of the verifier’s choosing (a problem that would require $\Theta(N)$ queries without access to a prover). That is, on a YES instance, if the dishonest verifier sent to the prover a range element k of its choosing (rather than a uniform random range element as the honest verifier does), then the prover will reply with a pre-image of k . The verifier would not have been able to compute such a pre-image on its own in $o(N)$ time, except with probability $o(1)$.

Chapter 11

Σ -Protocols and Commitments from Hardness of Discrete Logarithm

11.1 Cryptographic Background

11.1.1 A Brief Introduction to Groups

Informally, a group \mathbb{G} is any set equipped with an operation that behaves like multiplication. To be more precise, a group is a collection of elements equipped with a binary operation (which we denote by \cdot and refer to in this manuscript as multiplication) that satisfies the following four properties.

- Closure: the product of two elements in \mathbb{G} are also in \mathbb{G} , i.e., for all $a, b \in \mathbb{G}$, $a \cdot b$ is also in \mathbb{G} .
- Associativity: for all $a, b, c \in \mathbb{G}$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Identity: there an element denoted $1_{\mathbb{G}} \in \mathbb{G}$ such that $1_{\mathbb{G}} \cdot g = g \cdot 1_{\mathbb{G}} = g$ for all $g \in \mathbb{G}$.
- Invertibility: For each $g \in \mathbb{G}$, there is an element h in \mathbb{G} such that $g \cdot h = 1_{\mathbb{G}}$. This element h is denoted g^{-1} .

One important example of a group is the set of nonzero elements of any field, which forms a group under the field multiplication operation. This is referred to as the multiplicative group of the field. Another is the set of invertible matrices, which forms a group under the matrix multiplication operation.

Note that matrix multiplication is not commutative. In cases where the group operation *is* commutative, the group is called abelian.

Sometimes it is convenient to think of the group operation as addition rather than multiplication, in which case the operation is denoted with a $+$ symbol instead of \cdot . Whether a group is considered additive or multiplicative is a matter of context, convenience, and convention. As an example, any field is a group under the field's addition operation, and any the set of all $n \times n$ matrices over the field form a group under the matrix addition operation. For these groups it is of course natural to denote the group operation with $+$ rather than \cdot . Henceforth in this manuscript, *with the lone exception of two subsections in the Chapter 13 (Sections 13.1.4 and 13.1.5)*, we will exclusively refer to multiplicative groups, using \cdot to denote the group operation.

A *subgroup* of a group \mathbb{G} is a subset \mathbb{H} of \mathbb{G} that itself forms a group under the same binary operation as \mathbb{G} itself.

A group \mathbb{G} is said to be *cyclic* if there is some group element g such that all group elements can be generated by repeatedly multiplying g with itself, i.e., if every element of \mathbb{G} can be written as g^i for some positive integer i . Here, in analogy to how exponentiation refers to repeated multiplication in standard arithmetic, g^i denotes $\underbrace{g \cdot g \cdots g}_{i \text{ copies of } g}$.¹¹² Such an element of g is called a generator for \mathbb{G} . Any cyclic group is abelian.

The cardinality $|\mathbb{G}|$ is called the *order* of \mathbb{G} . A basic fact from group theory is that for any element $g \in \mathbb{G}$, $g^{|\mathbb{G}|} = 1_{\mathbb{G}}$. This implies that when considering any group exponentiation, i.e., g^ℓ for some integer ℓ , reducing the exponent ℓ modulo the group size $|\mathbb{G}|$ does not change anything. That is, for any integer ℓ , if $z \equiv \ell \pmod{|\mathbb{G}|}$, then $g^\ell = g^z$. Another basic fact from group theory states that the order of any subgroup \mathbb{H} of \mathbb{G} divides the order of \mathbb{G} itself.

11.1.2 The Discrete Logarithm Problem

For a specified group \mathbb{G} the discrete logarithm problem takes as input two group elements g and h , and the goal is to output a positive integer i such that $g^i = h$ (if \mathbb{G} is cyclic then such an i is guaranteed to exist).

The discrete logarithm problem is believed to be computationally intractable in certain groups \mathbb{G} . In modern cryptography, the groups used are typically cyclic subgroups of groups defined via elliptic curves over finite fields, or the multiplicative group of integers modulo a very large prime p . An important caveat is that quantum computers can solve the discrete logarithm problem in polynomial time via Shor’s algorithm [Sho94]. Hence, cryptosystems whose security is based on the assumed hardness of the discrete logarithm problem are not post-quantum secure.

Though it is a fascinating and important topic, we will not go into great detail on elliptic curve cryptography in this manuscript, and restrict ourselves to the following comments. Any elliptic curve group is defined with respect to a finite field \mathbb{F} , and group elements correspond to pairs of points $(x, y) \in \mathbb{F} \times \mathbb{F}$ that satisfy an equation of the form $y^2 = x^3 + ax + b$ for field elements a and b . The fastest known classical algorithm to solve the Discrete Logarithm problem over most elliptic curve groups used in practice runs in time $O(\sqrt{|\mathbb{G}|})$.¹¹³ Under the assumption that these are in fact the fastest attacks possible, this means that to obtain “ λ bits of security” (meaning security against attackers running in time 2^λ , see Footnote 78 in Section 6.3.2.2), one should use an elliptic curve group of order $2^{2\lambda}$. For example, a popular elliptic curve called Curve25519, which is defined over field \mathbb{F} of size $2^{255} - 19$, defines a cyclic group of order close to 2^{252} ; hence, this group provides slightly less than 128 bits of security [Ber06].^{114,115} One reason Curve25519 is popular is efficiency of group operations: the computational bottleneck in multiplying elliptic curve group elements turns out to be performing multiplications in the field \mathbb{F} over which the elliptic curve is defined. Because $p = 2^{255} - 19$ is a power of two minus a small constant, multiplication in \mathbb{F} can be implemented more efficiently than if p did not have this form. As general (rough) guidance, the time cost of perform-

¹¹²Similarly, g^{-i} denotes the i th power of the inverse of g , i.e., $(g^{-1})^i$.

¹¹³See, for example, the wikipedia article on Pollard’s rho algorithm https://en.wikipedia.org/wiki/Pollard%27s_rho_algorithm_for_logarithms, introduced in [Pol78].

¹¹⁴Note that the size of the field \mathbb{F} over which the elliptic curve is defined is not the same as the order $|\mathbb{G}|$ of the elliptic curve group, but the field size $|\mathbb{F}|$ and group size $|\mathbb{G}|$ are tightly related: a result known as Hasse’s theorem [Has36] states that for all elliptic curve groups \mathbb{G} over field \mathbb{F} , $|\mathbb{G}| - (|\mathbb{F}| + 1) \leq 2\sqrt{|\mathbb{F}|}$.

¹¹⁵A subtlety arising in modern elliptic curves used in cryptography is that the group order is typically a small constant—typically 4 or 8—times a prime. For example, the order of Curve25519 is 8 times a prime. For this reason, implementations typically work over the prime-order subgroup of the full elliptic curve group, or they add a layer of abstraction that exposes a prime-order interface. The interested reader is directed to [Ham15] and https://ristretto.group/why_ristretto.html for an overview of these details.

ing one group multiplication in an elliptic curve group defined over field \mathbb{F} is typically about 10 times as expensive as performing one multiplication in \mathbb{F} .

11.2 Schnorr’s Σ -Protocol for Knowledge of Discrete Logarithms

In this section, we describe several perfect honest-verifier zero-knowledge proof systems. These proof systems have a very simple structure, involving only three messages exchanged between prover and verifier. They are *special-purpose*, meaning that, as standalone objects, they do not solve **NP**-complete problems such as circuit satisfiability. Rather, they solve specific problems including (a) establishing that the prover has knowledge of a discrete logarithm of some group element (Section 11.2.2); (b) allowing the prover to cryptographically commit to group elements without revealing the committed group element to the verifier until later (Section 11.3); and (c) establishing product relationships between committed values (Section 11.3.2).

While the protocols covered in this section are special-purpose, we will see (e.g., Section 12.1) that they can be *combined* with general-purpose protocols such as IPs, IOPs, and MIPs to obtain general-purpose zk-SNARKs.

11.2.1 Σ -Protocols

The presentation in this section closely follows other authors [BLAG19]. A relation \mathcal{R} specifies a collection of “valid” instance-witness pairs (h, w) . For example, given a group \mathbb{G} and a generator g , the discrete logarithm relation $\mathcal{R}_{\text{DL}}(\mathbb{G}, g)$ is the collection of pairs $(h, w) \in \mathbb{G} \times \mathbb{Z}$ such that $h = g^w$.

A Σ -protocol for a relation \mathcal{R} is a 3-message public coin protocol between prover and verifier in which both prover and verifier know a public input h , and the prover knows a witness w such that $(h, w) \in \mathcal{R}$.¹¹⁶ Let us denote the three messages by (a, e, z) , with the prover first sending a , the verifier responding with a challenge e (chosen via public random coin tosses), and the prover replying with z . A Σ -protocol is required to satisfy perfect completeness, i.e., if the prover follows the prescribed protocol then the verifier will accept with probability 1. It is also required to satisfy two additional properties.

Special soundness: There exists a polynomial time algorithm \mathcal{Q} such that, when given as input a pair of accepting transcripts (a, e, z) and (a, e', z') with $e \neq e'$, \mathcal{Q} outputs a witness w such that $(h, w) \in \mathcal{R}$.

Intuitively, special soundness guarantees that if, after sending its first message in the Σ -protocol, the prover is prepared to answer more than one challenge from the verifier, then the prover must *know* a witness w such that $(h, w) \in \mathcal{R}$.

Honest Verifier Perfect Zero-Knowledge. There must be a randomized polynomial time simulator that takes as input the public input h from the Σ -protocol, and outputs a transcript (a, e, z) such that the distribution over transcripts output by the simulator is *identical* to the distribution over transcripts produced by the honest verifier in the Σ -protocol interacting with the honest prover.

Remark 11.1. Special soundness implies that, if the verifier in the Σ -protocol were to be given “rewinding access” to the prover, then the Σ -protocol would *not* be zero-knowledge. That is, special soundness says that if the verifier could run the protocol to completion to obtain the transcript (a, e, z) , then “rewind” to just

¹¹⁶The term Σ -protocol was coined because pictorial diagrams of 3-message protocols are vaguely reminiscent of the Greek letter Σ .

after the prover sent its first message a , and restart the protocol with a new challenge e' , then (assuming both transcripts lead to acceptance), the verifier would learn a witness. This clearly violates zero-knowledge if witnesses are assumed to be intractable to compute. Hence, the honest-verifier zero-knowledge property of Σ -protocols only holds if the verifier is never allowed to run the prover more than once with the same first prover message a .

11.2.2 Schnorr's Σ -Protocol for the Discrete Logarithm Relation

Let \mathbb{G} be a cyclic group of prime order generated by g . Recall that in any Σ -protocol for the discrete logarithm relation, \mathcal{P} holds (h, w) such that $h = g^w$ in \mathbb{G} , while \mathcal{V} knows h and g .¹¹⁷

To convey the intuition behind Schnorr's [Sch89] protocol, we describe a number of progressively more sophisticated attempts at designing a proof of knowledge for the discrete logarithm relation.

Attempt 1. The most straightforward possible proof of knowledge for any relation is to simply have the prover \mathcal{P} send the witness w for the public input h , so the verifier \mathcal{V} can check that $(h, w) \in \mathcal{R}$. However, this reveals w to the verifier, violating zero-knowledge (assuming the verifier could not efficiently compute the witness on her own).

Attempt 2. \mathcal{P} could pick a random value $r \in \{0, \dots, |\mathbb{G}| - 1\}$ and send $(w + r) \bmod |\mathbb{G}|$ to \mathcal{V} . This totally "hides" w in that $(w + r) \bmod |\mathbb{G}|$ is a uniform random element of the set $\{0, \dots, |\mathbb{G}| - 1\}$, and hence this message does not violate zero-knowledge. But for the same reason, $(w + r) \bmod |\mathbb{G}|$ is useless to \mathcal{V} as far as ensuring soundness goes. It is simply a random number, which \mathcal{V} could have generated on her own.

Attempt 3. To address the issue in Attempt 2 that $(w + r) \bmod |\mathbb{G}|$ is useless on its own, \mathcal{P} could first send r , followed by a value z claimed to equal $(w + r) \bmod |\mathbb{G}|$. \mathcal{V} checks that $g^r \cdot h = g^z$.

This protocol is complete and sound, but it is not zero-knowledge. Completeness is easy to check, while special soundness holds because if $g^r \cdot h = g^z$, then $g^{z-r} = h$, i.e., $z - r$ is a witness. That is, a witness can be extracted from even a single accepting transcript. Of course, for the same reason, this protocol is not zero-knowledge.

Effectively, Attempt 3 broke w into two pieces, $z := w + r \bmod |\mathbb{G}|$ and r , such that each piece individually reveals no information to \mathcal{V} (because each is simply a random element of $\{0, 1, \dots, |\mathbb{G}|\}$). But together, the pieces reveal the witness w to the verifier (since $z - r = w$). Hence, this attempt is no closer to satisfying zero-knowledge than Attempt 1.

Attempt 4. We could modify Attempt 3 above so that, rather than \mathcal{P} sending r to \mathcal{V} , \mathcal{P} instead sends a group element a claimed to equal g^r , followed by a number z exactly as in Attempt 3, i.e., z is claimed to equal $(w + r) \bmod |\mathbb{G}|$. \mathcal{V} checks that $a \cdot h = g^z$.

This attempt turns out to be complete and zero-knowledge, but not special sound. Completeness is easy to verify: if the prover is honest, then $a \cdot h = g^r \cdot h = g^{r+w} = g^z$. It is zero-knowledge because a simulator can choose an element $z \in \{0, 1, \dots, |\mathbb{G}| - 1\}$ at random, and then set a to be $g^z \cdot h^{-1}$, and output the transcript (a, z) . This generates a transcript distributed identically to that generated by the honest prover.

Conceptually, while the honest prover in Attempt 4 chooses a random group element $a = g^r$ and then chooses z to be the unique number such that the verifier accepts (a, z) , the simulator chooses z first at

¹¹⁷In this manuscript we only consider Σ -protocols for groups of prime order. Σ -protocols (and related proof systems) for problems over composite and hidden-order groups have also been studied, see for example [FO97, BCK10, BBF19].

random and then chooses a to be the unique group element causing the verifier to accept (a, z) . The two distributions are identical—in both cases, a and z are individually uniformly distributed (a from \mathbb{G} and z from $\{0, 1, \dots, |\mathbb{G}| - 1\}$), with the value of a determining the value of z and vice versa.

Sadly, Attempt 4 is not special sound for the same reason it is zero-knowledge. The simulator is able to generate accepting transcripts, and since the protocol is totally non-interactive (there is no challenge sent by verifier to prover), the simulator itself acts as a “cheating” prover capable of convincing the verifier to accept despite not knowing a witness.

Comparison of Attempts 3 and 4. The reason Attempt 4 is zero-knowledge while Attempt 3 is not is that whereas Attempt 3 has \mathcal{P} send r “in the clear”, Attempt 4 “hides” r in the exponent of g , and accordingly the subtraction of r from z by the verifier in Attempt 4 happens “in the exponent” of g rather than in the clear.

The fact that Attempt 4 is zero-knowledge may seem surprising at first. After all, on an information-theoretic level, r can be derived from g^r , and then the witness $z - r$ can be computed, and this may seem like a violation of zero-knowledge. But the derivation of r requires finding the discrete logarithm of g^r , which is just as hard as deriving a witness w (i.e., a discrete logarithm of h). In summary, the fact that Attempt 4 reveals r to the verifier in an information-theoretic sense does not contradict zero-knowledge, because the public input $h = g^w$ itself information-theoretically specifies w in the same way that $a = g^r$ information-theoretically specifies r . In fact, g^r combined with $(w + r) \bmod |\mathbb{G}|$ does not actually reveal any new information to the verifier beyond what was already revealed by h itself.

Schnorr’s Σ -Protocol. Protocol 3 describes Schnorr’s Σ -protocol. Essentially, Schnorr’s protocol modifies Attempt 4 so that, after \mathcal{P} sends a but before \mathcal{P} sends z , the verifier sends a random challenge e drawn from $\{0, 1, \dots, |\mathbb{G}| - 1\}$. Compared to Attempt 4, the verifier’s check is modified so that it will pass if $z = w \cdot e + r$ (Attempt 4 above is identical to Schnorr’s protocol with the verifier’s challenge e fixed to 1).

These modifications to Attempt 4 do not alter the completeness or zero-knowledge properties of the protocol. The intuition for why Schnorr’s protocol is special sound is that if \mathcal{P} ’s first message is $a = g^r$ and \mathcal{P} can produce accepting transcripts (a, e, z) and (a, e', z') with $e \neq e'$, then \mathcal{V} ’s acceptance criterion implies that $z = w \cdot e + r$ and $z' = w \cdot e' + r$.¹¹⁸ These are two linearly independent equations in two unknowns, namely w and r . Hence, one can take these two transcripts and efficiently solve for both w and r .

Protocol 3 Schnorr’s Σ -protocol for the Discrete Logarithm Relation

- 1: Let \mathbb{G} be a (multiplicative) cyclic group of prime order with generator g .
 - 2: Public input is $h = g^w$, where only prover knows w .
 - 3: \mathcal{P} picks a random number r in $\{0, \dots, |\mathbb{G}| - 1\}$ and sends $a \leftarrow g^r$ to the verifier.
 - 4: Verifier responds with a random element $e \in \{0, \dots, |\mathbb{G}| - 1\}$.
 - 5: Prover responds with $z \leftarrow (we + r) \bmod |\mathbb{G}|$.
 - 6: Verifier checks that $a \cdot h^e = g^z$.
-

We now turn to formally proving that Schnorr’s protocol satisfies perfect completeness, special soundness, and honest-verifier zero-knowledge.

Perfect completeness is easy to establish: if $a \leftarrow g^r$ and $z \leftarrow (we + r) \bmod |\mathbb{G}|$, then

$$a \cdot h^e = g^r \cdot h^e = g^r \cdot (g^w)^e = g^{r+we} = g^z,$$

¹¹⁸Similar to Attempt 4, note that \mathcal{V} ’s check on transcript (a, e, z) in Schnorr’s protocol confirms “in the exponent” that $z = w \cdot e + r$. \mathcal{V} is able to perform this check in the exponent despite only knowing z , g^r , and h (in particular, without knowing r and w , which are the discrete logarithms of g^r and h).

so the verifier accepts transcript (a, e, z) .

Special soundness: Suppose we are given two accepting transcripts (a, e, z) and (a, e', z') with $e \neq e'$. We must show that a witness w^* can be extracted in polynomial time from these two transcripts.

Let $(e - e')^{-1}$ denote the multiplicative inverse of $e - e'$ modulo $|\mathbb{G}|$, i.e., $(e - e')^{-1}$ denotes a number ℓ such that $\ell \cdot (e - e') \equiv 1 \pmod{|\mathbb{G}|}$. Since $e \neq e'$, such a multiplicative inverse is guaranteed to exist because $|\mathbb{G}|$ is prime and every nonzero number has a multiplicative inverse modulo any prime, and in fact ℓ can be computed efficiently via the Extended Euclidean algorithm.

Let $w^* = ((z - z') \cdot (e - e')^{-1}) \pmod{|\mathbb{G}|}$. To see that w^* is a witness, observe that since (a, e, z) and (a, e', z') are both accepting transcripts, it holds that $a \cdot h^e = g^z$ and $a \cdot h^{e'} = g^{z'}$. Since \mathbb{G} is cyclic and g is a generator of \mathbb{G} , both a and h are powers of g , say, $a = g^j$ and $h = g^w$ for integers j, w . Then the preceding two equations imply that

$$\begin{aligned} g^{j+we} &= g^z \\ g^{j+we'} &= g^{z'}. \end{aligned}$$

Together, these two equations imply that

$$g^{w(e-e')} = g^{z-z'}.$$

Hence, $w(e - e') \equiv z - z' \pmod{|\mathbb{G}|}$, i.e., $w \equiv (z - z') \cdot (e - e')^{-1} \pmod{|\mathbb{G}|} = w^*$. That is, $h^w = h^{w^*}$, meaning that w^* is a witness.

Honest-Verifier Perfect Zero Knowledge. We need to construct a polynomial time simulator that produces a distribution over transcripts (a, e, z) identical to the distribution produced by the honest verifier and prover. The simulator selects e uniformly at random from $\{0, \dots, |\mathbb{G}| - 1\}$ and samples z uniformly at random from $\{0, \dots, |\mathbb{G}| - 1\}$. Finally, the simulator sets $a \leftarrow g^z \cdot (h^e)^{-1}$.

The distribution over transcripts produced by the simulator is identical to that produced by the honest verifier interacting with the prescribed prover. In both cases, the distribution produces a random $e \in \{0, \dots, |\mathbb{G}| - 1\}$, and then chooses a pair (a, z) such that a is chosen uniformly random from \mathbb{G} and z from $\{0, \dots, |\mathbb{G}| - 1\}$, subject to the constraint that $a \cdot h^e = g^z$ (the key observation from which this follows is that, for fixed e , for any $a \in \mathbb{G}$ there is exactly one $z \in \{0, \dots, |\mathbb{G}| - 1\}$ satisfying this equality, and vice versa).

Remark 11.2. Schnorr's protocol is only *honest-verifier* zero knowledge (HVZK) because the simulated distribution over transcripts is identical to the verifier's view in the actual protocol only if the verifier's message e is a uniformly random element from $\{0, \dots, |\mathbb{G}| - 1\}$. Two remarks are in order. First, if we render the protocol non-interactive using the Fiat-Shamir transformation (see Section 11.2.3), the distinction between honest-verifier and dishonest-verifier zero-knowledge is eliminated (see Footnote 103 for a brief discussion of this point). Second, it turns out that Schnorr's protocol actually *is* dishonest-verifier zero-knowledge, with the following caveat: the simulation is efficient only if the challenge e is not selected at random from $\{0, \dots, |\mathbb{G}| - 1\}$, but rather is only permitted to be selected from a designated polynomial-size subset S of \mathbb{G} (this is because the known simulator for an arbitrary dishonest verifier's view has a runtime that grows with $|S|$). To obtain negligible soundness error from such a protocol, one must repeat it $\omega(1)$ many times sequentially, adding additional communication and computation costs. The interested reader is directed to [Mau09, Section 4] for details.

11.2.3 Fiat-Shamir Applied to Σ -Protocols

In this section, we explain that applying the Fiat-Shamir transformation (Section 4.7.2) to any Σ -protocol (such as Schnorr’s) yields a non-interactive argument of knowledge in the random oracle model. This result is originally due to Pointcheval and Stern [PS00].

For concreteness, we couch the presentation in the context of Schnorr’s protocol, where the input is a group element h , and the prover claims to know a witness w such that $h = g^w$, where g is a specified group generator. Recall that in the resulting non-interactive argument, the honest prover aims to produce an accepting transcript (a, e, z) for the Σ -protocol, where $e = R(h, a)$ and R denotes the random oracle.

Let \mathcal{I} refer to the Σ -protocol and \mathcal{Q} refer to the non-interactive argument obtained by applying the Fiat-Shamir transformation to \mathcal{I} . Let $\mathcal{P}_{\mathcal{FS}}$ be a prover for \mathcal{Q} that produces a convincing proof on input h with probability at least ε . That is, when $\mathcal{P}_{\mathcal{FS}}$ is run on input h , it outputs a transcript (a, e, z) that, with probability at least ε , is an accepting transcript for \mathcal{I} and satisfies $e = R(h, a)$ (here, the probability is over the choice of random oracle and any internal randomness used by $\mathcal{P}_{\mathcal{FS}}$). We show that by running $\mathcal{P}_{\mathcal{FS}}$ at most twice, we can, with probability at least $\Omega(\varepsilon^4/T^3)$, “extract” from $\mathcal{P}_{\mathcal{FS}}$ two accepting transcripts for \mathcal{I} of the form (a, e, z) and (a, e', z') with $e \neq e'$.¹¹⁹ By special soundness of \mathcal{I} , these two transcripts can in turn be efficiently transformed into a witness w . If T is polynomial and ε is non-negligible, then $\Omega(\varepsilon^4/T^3)$ is non-negligible, contradicting the assumed intractability of finding a witness.¹²⁰

What we can assume about $\mathcal{P}_{\mathcal{FS}}$ without loss of generality. As in the proof of Theorem 4.9, we will assume that $\mathcal{P}_{\mathcal{FS}}$ always makes exactly T queries to the random oracle R , that all queries $\mathcal{P}_{\mathcal{FS}}$ makes are distinct, and that $\mathcal{P}_{\mathcal{FS}}$ always outputs a transcript of the form (a, e, z) with $e = R(h, a)$, such that at least one of $\mathcal{P}_{\mathcal{FS}}$ ’s T queries to R was at point (h, a) . See the proof of Theorem 4.9 for an explanation of why these assumptions are without loss of generality.

The witness extraction procedure. There is a natural way to extract from $\mathcal{P}_{\mathcal{FS}}$ two accepting transcripts (a, e, z) and (a, e', z') . First, fix the value of any internal randomness used by $\mathcal{P}_{\mathcal{FS}}$. The first transcript is obtained by simply running $\mathcal{P}_{\mathcal{FS}}$ once, generating a random value for R ’s response to each query $\mathcal{P}_{\mathcal{FS}}$ makes to the random oracle. This yields a transcript (a, e, z) satisfying $e = R(h, a)$ such that with probability at least ε the transcript is an accepting one for \mathcal{I} . By assumption, during this execution of $\mathcal{P}_{\mathcal{FS}}$, exactly one of the T queries to R was equal to (h, a) . Rewind $\mathcal{P}_{\mathcal{FS}}$ to just before it queries R at (h, a) , and change R ’s response to this query from e to a fresh randomly chosen value e' . Then run $\mathcal{P}_{\mathcal{FS}}$ once again to completion (again generating a random value from R ’s response to each query made by $\mathcal{P}_{\mathcal{FS}}$), and hope that $\mathcal{P}_{\mathcal{FS}}$ outputs an accepting transcript of the form (a, e', z') .

Analysis of the witness extraction procedure. We must show that the probability this procedure outputs two accepting transcripts of the form (a, e, z) and (a, e', z') with $e \neq e'$ is at least $\Omega(\varepsilon^3/T^2)$. Note that e will not equal e' with probability $1 - 1/2^\lambda$, where λ denotes the number of bits output by R on any query. For simplicity, let us assume henceforth that $e \neq e'$, as this will affect the success probability of the extraction procedure by at most $1/2^\lambda$.

Key to the analysis is the following basic result in probability theory.

¹¹⁹For simplicity, we do not provide a quantitatively tight analysis of the witness extraction procedure.

¹²⁰One can find a witness with constant probability instead of just with non-negligible probability by running the witness-finding procedure $\ell = O(T^3/\varepsilon^4)$ times. The probability that all ℓ invocations of the procedure fail to find a witness is at most $(1 - 1/\ell)^\ell \leq 1/e < 1/2$.

Claim 11.1. Suppose (X, Y) are jointly distributed random variables and let $A(X, Y)$ be any event such that $\Pr[A(X, Y)] \geq \epsilon'$. Let μ_X be the marginal distribution of X , and for x in the support of μ_X , call x good if the conditional probability $\Pr[A(X, Y)|X = x]$ is at least $\epsilon'/2$. Let $p = \Pr_{x \sim \mu_X}[x \text{ is good}]$. Then $p \geq \epsilon'/2$.

Proof. If x is not good, let us call x bad. We can write:

$$\begin{aligned} \Pr[A(X, Y) \geq \epsilon'] &= \Pr[A(X, Y) \geq \epsilon' | X \text{ is good}] \cdot \Pr_{\mu_X}[X \text{ is good}] + \Pr[A(X, Y) \geq \epsilon' | X \text{ is bad}] \cdot \Pr_{\mu_X}[X \text{ is bad}] \\ &\leq 1 \cdot p + \epsilon'/2. \end{aligned}$$

Since $\Pr[A(X, Y) \geq \epsilon']$, we conclude that $p \geq \epsilon'/2$. \square

Say that $\mathcal{P}_{\mathcal{FS}}$ wins if the transcript (a, e, z) that $\mathcal{P}_{\mathcal{FS}}$ produces is an accepting one satisfying $e = R(h, a)$. Consider applying Claim 11.1, with X equal to $\mathcal{P}_{\mathcal{FS}}$'s internal randomness, Y equal to the evaluations of the random oracle R , and A equal to the event that $\mathcal{P}_{\mathcal{FS}}$ wins when run with internal randomness X and random oracle Y . Claim 11.1 implies that with probability at least $\epsilon/2$, $\mathcal{P}_{\mathcal{FS}}$'s internal randomness is "good", which in this context means that when the internal randomness is set to X , the probability over the random oracle R that $\mathcal{P}_{\mathcal{FS}}$ produces an accepting transcript (a, e, z) with $e = R(h, a)$ is at least $\epsilon/2$. Let E be the event that $\mathcal{P}_{\mathcal{FS}}$'s internal randomness is good. We can write the probability that the witness extraction procedure succeeds as

$$\Pr[E] \cdot \Pr[\text{witness extraction succeeds} | E] \geq (\epsilon/2) \cdot \Pr_R[\text{witness extraction succeeds} | E].$$

For the remainder of the proof, we bound $\Pr_R[\text{witness extraction succeeds} | E]$. For notational brevity, we will leave the conditioning on E implicit when writing out the probabilities of various events. By conditioning on E , we may henceforth consider $\mathcal{P}_{\mathcal{FS}}$ to be a *deterministic* algorithm (i.e., no internal randomness), that wins with probability at least $\epsilon/2$ over the random choice of the random oracle R .

Let Q_1, \dots, Q_T denote the T queries that $\mathcal{P}_{\mathcal{FS}}$ makes to the random oracle (note that these are random variables that depend on R). Next, we claim that there is at least one integer $i^* \in \{1, \dots, T\}$ such that

$$\Pr_R[\mathcal{P}_{\mathcal{FS}} \text{ wins} \cap Q_{i^*} = (h, a)] \geq \epsilon/(2T). \quad (11.1)$$

Indeed, if $\Pr_R[\mathcal{P}_{\mathcal{FS}} \text{ wins} \cap Q_i = (h, a)] < \epsilon/(2T)$ for all $i = 1, \dots, T$, then since we have assumed that for any transcript (a, e, z) output by $\mathcal{P}_{\mathcal{FS}}$ there is some $i \in \{1, \dots, T\}$ such that $Q_i = (h, a)$,

$$\Pr_R[\mathcal{P}_{\mathcal{FS}} \text{ wins}] \leq \sum_{i=1}^T \Pr_R[\mathcal{P}_{\mathcal{FS}} \text{ wins} \cap Q_i = (h, a)] < T \cdot (\epsilon/2) = \epsilon/2,$$

a contradiction.

Let i^* satisfy Equation (11.1). Consider applying Claim 11.1, with X equal to R 's responses to the first $i^* - 1$ queries, and Y equal to R 's responses to the remaining $T - i^* + 1$ queries. Let A be the event that $\mathcal{P}_{\mathcal{FS}}$, when run with random oracle R , produces a winning transcript (a, e, z) with (h, a) equal to $\mathcal{P}_{\mathcal{FS}}$'s (i^*) 'th query, namely Q_{i^*} .

For a value of x in the support of X , call x good if $A(x, Y)$ holds with probability at least $\epsilon/(4T)$ over the choice of Y . Equation (11.1) asserts that $\Pr_{X, Y}[A(X, Y)] \geq \epsilon/(2T)$. Hence, Claim 11.1 asserts that X is good with probability at least $\epsilon/(4T)$.

We can think of the process of generating the two transcripts (a, e, z) and (a', e', z') as first selecting X (thereby determining the first i^* queries Q_1, \dots, Q_{i^*} made by $\mathcal{P}_{\mathcal{FS}}$), then drawing two independent copies Y'

and Y'' of Y . Both (a, e, z) and (a', e', z') are accepting transcripts with $a = a' = Q_{i^*}$ if (X, Y) and (X', Y') both satisfy event A . This probability is at least

$$\Pr_{x \sim X}[x \text{ is good}] \cdot \Pr_Y[A(x, Y) | x \text{ is good}] \cdot \Pr_{Y'}[A(x, Y') | x \text{ is good}] \geq (\varepsilon / (4T))^3.$$

In conclusion (taking into account that the argument above has conditioned on the event E that the choice of $\mathcal{P}_{\mathcal{FS}}$'s internal randomness is good, an event that happens with probability at least $\varepsilon/2$), we have shown our witness-extraction procedure succeeds with probability at least $\Omega(\varepsilon^4/T^3)$ as claimed.

Remark 11.3. Results lower bounding the success probability of witness extraction procedures related to the one in this section are called *forking lemmas*. The terminology highlights the fact that the witness extraction procedure runs $\mathcal{P}_{\mathcal{FS}}$ twice, once using random oracle responses (X, Y) and once using (X, Y') , where X captures the random oracle's responses to the first i^* queries made by $\mathcal{P}_{\mathcal{FS}}$ and Y and Y' capture responses to the remaining queries. One thinks of the random oracle generation process as “forking” into two different paths after the first i^* responses are generated.

11.3 A Homomorphic Commitment Scheme

Commitment Schemes. In a commitment scheme, there are two parties, a committer and a verifier. The committer wishes to bind itself to a message without revealing the message to the verifier. That is, once the committer sends a commitment to some message m , it should be unable to “open” to the commitment to any value other than m (this property is called binding). But at the same time the commitment itself should not reveal information about m to the verifier (this is called hiding).

Most properties come in statistical and computational flavors, just like soundness in interactive proofs and arguments. That is, binding can hold statistically (meaning even computationally unbounded committers are unable to open a commitment to two different messages, except with negligible probability of success) or computationally (only polynomial-time committers are unable to open commitments to two different messages). Similarly, hiding may be statistical (even computationally unbounded verifiers cannot extract any information about m from the commitment to m) or computational (only polynomial time verifiers are unable to extract information about m from the commitment). A commitment can be statistically binding and computationally hiding or vice versa, but it cannot be simultaneously statistically hiding and binding, as any commitment that statistically binds the committer to a message must by definition reveal the message in a statistical sense.¹²¹ In this manuscript, we will only consider commitment schemes that are computationally binding and perfectly hiding.

Formally, a commitment scheme is specified by three algorithms, KeyGen, Commit, and Verify. KeyGen is a randomized algorithm that generates a commitment key ck and verification key vk that are available to the committer and the verifier respectively (if all keys are public then $ck = vk$), while Commit is a randomized algorithm that takes as input the committing key ck and the message m to be committed and outputs the commitment c , as well as possibly extra “opening information” d that the committer may hold onto and only reveal during the verification procedure. Verify takes as input the commitment, the verification key, and a claimed message m' provided by the committer, and any opening information d and decides whether to accept m' as a valid opening of the commitment.

¹²¹A computationally unbounded verifier could simulate a computationally unbounded cheating prover's efforts to open the commitment to multiple messages; statistical binding guarantees that these efforts will succeed for only one message except with negligible probability.

A commitment scheme is *correct* if $\text{Verify}(\text{vk}, \text{Commit}(m, \text{ck}), m)$ accepts with probability 1, for any m (i.e., an honest committer can always successfully open the commitment to the value that was committed). A commitment scheme is perfectly hiding if the distribution of the commitment $\text{Commit}(m, \text{ck})$ is independent of m . Finally, a commitment scheme is computationally binding if for every polynomial time algorithm \mathcal{Q} , the probability of winning the game depicted in Protocol 4 is negligible (i.e., inverse-superpolynomial in the security parameter).

Protocol 4 Binding Game for Commitment Schemes

- 1: $(\text{vk}, \text{ck}) \leftarrow \text{KeyGen}()$
 - 2: $(c, d, m, d', m') \leftarrow \mathcal{Q}(\text{ck})$
 - ▷ c should be thought of as a commitment.
 - ▷ d and d' should be thought of as opening information, to open c to messages m and m' respectively.
 - 3: \mathcal{Q} wins if $\text{Verify}(\text{vk}, (c, d), m) = \text{Verify}(\text{vk}, (c, d'), m') = 1$ and $m \neq m'$
-

A Perfectly Hiding Commitment Scheme from any Σ -Protocol. Informally, a relation \mathcal{R} is said to be hard if there is no efficient algorithm for identifying a witness w such that $(h, w) \in \mathcal{R}$. More precisely, a *hard relation* is one for which there is some efficient randomized algorithm Gen that generates “hard instances” of the relation in the following sense. Gen outputs pairs (h, w) , and there is no polynomial time algorithm that, when fed the value h output by Gen , can find a witness w' such that $(h, w') \in \mathcal{R}$ except with negligible probability. For example, for the discrete logarithm relation in prime order groups \mathbb{G} with generator g for which the discrete logarithm problem is believed to be intractable, Gen would pick a random integer $r \in \{0, \dots, |\mathbb{G}| - 1\}$ and output (h, r) where $h = g^r$.

Damgård [Dam89] showed how to use any Σ -protocol for any hard relation to obtain a perfectly hiding, computationally binding commitment scheme. By instantiating Damgård’s construction with Shnorr’s Σ -protocol [Sch89] for the discrete logarithm relation, one recovers a well-known commitment scheme due to Pedersen [Ped91] that will play an important role in this manuscript. (The typical presentation of Pedersen’s commitment scheme differs slightly, in an entirely cosmetic manner, from the version recovered here. See Protocols 5 and 6 for details.)

Actually, to ensure hiding, Damgård’s transformation does require the Σ -protocol to satisfy one property that was not mentioned above. The simulator used to establish HVZK must be able to take as input not only the public input h , but also a challenge e^* , and output a transcript (a, e^*, z) such that the distribution over transcripts produced by the simulator is identical to the distribution over transcripts produced by the interaction of the verifier and prescribed prover when the verifier’s challenge is fixed to e^* . This property is called *special honest-verifier perfect zero-knowledge*. The simulator for Schnorr’s Σ -protocol satisfies this property simply by fixing the challenge chosen by the simulator to e^* , rather than having the simulator choose the challenge at random from the challenge space.

Here is how Damgård’s commitment scheme works. The key generation procedure runs the generation algorithm for the hard relation \mathcal{R} to obtain an (instance, witness) pair $(h, w) \leftarrow \text{Gen}$, and declares h to be both the committing key ck and the verification key vk . Note that the witness w represents “toxic waste” that must be discarded, in the sense that anyone who knows w may be able to break binding of the commitment scheme. To commit to a message m , the committer runs the simulator from the Σ -protocol for \mathcal{R} (whose existence is guaranteed by the special HVZK property of the Σ -protocol) on public input h to generate a transcript in which the challenge is the message m (this is where the property of the simulator described in the previous paragraph is exploited). Let (a, e, z) be the output of the simulator. The committer sends a as the commitment, and keeps $e = m$ and z as opening information. In the verification stage for the commitment

scheme, the committer sends the opening information $e = m$ and z to the verifier, who uses the verification procedure of the Σ -protocol to confirm that (a, e, z) is an accepting transcript for public input h .¹²²

We need to show that the commitment scheme satisfies correctness, computational binding, and perfect hiding. Correctness is immediate from the fact that the HVZK property of the Σ -protocol guarantees that the simulator only outputs accepting transcripts. Perfect hiding follows from the fact that in any Σ -protocol, the first message a sent by the prover is independent of the verifier’s challenge in the Σ -protocol (which equals the message being committed to in the commitment scheme). Computational binding follows from special soundness of the Σ -protocol: if the committer could output a commitment a and two sets of “opening information” (e, z) and (e', z') that both cause the commitment verifier to accept, then (a, e, z) and (a, e', z') must be accepting transcripts for the Σ -protocol, and there is an efficient procedure to take two such transcripts and produce a witness w such that $(h, w) \in \mathcal{R}$. The fact that \mathcal{R} is hard means that this can only be done with non-negligible probability if the committer runs in superpolynomial time.

Note that when applying the transformation to Schnorr’s protocol for the discrete logarithm relation, the key generation procedure produces a random power of generator g , which is simply a random group element h . Hence, the commitment key and verification key in the resulting commitment scheme can be generated *transparently* (meaning no toxic waste produced). That is, rather than choosing a witness r at random letting $h = g^r$ (producing toxic waste r which could be used to break binding of the commitment scheme), h can be directly chosen to be a random group element. In this way, no one knows the discrete logarithm of h to base g (and by assumption, computing this discrete logarithm given h and g is intractable).

The resulting commitment scheme is displayed in Protocol 6. The traditional (and equivalent, up to cosmetic differences) presentation of Pedersen commitments is given in Protocol 5 for comparison. To maintain consistency with the literature, for the remainder of this manuscript we follow the traditional presentation of Pedersen commitments (Protocol 5). In the traditional presentation, to commit to a message m , the committer picks a random exponent z in $\{0, \dots, |\mathbb{G}| - 1\}$ and the commitment is $g^m \cdot h^z$. One thinks of h^z as a random group element that operates as a “blinding factor”: by multiplying g^m by h^z , one ensures that the commitment is a random group element, statistically independent of m .¹²³

Protocol 5 Standard presentation of Pedersen commitments in a cyclic group \mathbb{G} for which the Discrete Logarithm problem is intractable.

- 1: Let \mathbb{G} be a (multiplicative) cyclic group of prime order. The key generation procedure publishes randomly chosen generators $g, h \in \mathbb{G}$, which serve as both the commitment key and verification key.
 - 2: To commit to a number $m \in \{0, \dots, |\mathbb{G}| - 1\}$, committer picks a random $z \in \{0, \dots, |\mathbb{G}| - 1\}$ and sends $c \leftarrow g^m \cdot h^z$.
 - 3: To open a commitment c , committer sends (m, z) . Verifier checks that $c = g^m \cdot h^z$.
-

11.3.1 Important Properties of Pedersen Commitments

Additive Homorphism. One important property of Pedersen commitments is that they are *additively homomorphic*. This means that the verifier can take two commitments c_1 and c_2 , to values $m_1, m_2 \in$

¹²²If the committed message m contains data that the verifier couldn’t compute on its own, then revealing m to the verifier violates zero-knowledge. In our actual zero-knowledge arguments that make use of Pedersen commitments, the prover will never actually open any commitment, but rather will prove in zero-knowledge that it *could* open the commitment if it wanted to. See Protocol 7.

¹²³The blinding factor h^z ensures that the Pedersen commitment is *perfectly* (i.e., statistically) hiding. If the blinding factor is omitted, the commitment is still computationally hiding. Roughly speaking, this is because computing m from the “unblinded” commitment g^m requires solving the discrete logarithm problem to base g , the intractability of which is already assumed by the binding analysis.

Protocol 6 Commitment scheme obtained from Schnorr’s protocol via Damgård’s transformation. This is the same as Protocol 5 except for the cosmetic difference that the commitment is taken to be $h^{-m} \cdot g^z$ instead of $g^m \cdot h^z$, with the verification procedure modified accordingly (i.e., the roles of g and h are reversed, and m is replaced with $-m$).

- 1: Let \mathbb{G} be a (multiplicative) cyclic group of prime order.
- 2: The key generation procedure publishes randomly chosen generators $g, h \in \mathbb{G}$, which serve as both the commitment key and verification key.
- 3: To commit to a number $m \in \{0, \dots, |\mathbb{G}| - 1\}$, committer picks a random $z \in \{0, \dots, |\mathbb{G}| - 1\}$ and sends

$$c \leftarrow h^{-m} \cdot g^z.$$

- 4: To open a commitment c , committer sends (m, z) . Verifier checks that $c \cdot h^m = g^z$.
-

$\{0, \dots, |\mathbb{G}| - 1\}$ (with m_1, m_2 known to the committer but not to the verifier), and the verifier on its own can derive a commitment c_3 to $m_3 := m_1 + m_2$, such that the prover is able to open c_3 to m_3 . This is done by simply letting $c_3 \leftarrow c_1 \cdot c_2$. As for “opening information” provided by the prover, if $c_1 = h^{m_1} \cdot g^{z_1}$ and $c_2 = h^{m_2} \cdot g^{z_2}$, then $c_3 = h^{m_1+m_2} \cdot g^{z_1+z_2}$, so the opening information for c_3 is simply $(m_1 + m_2, z_1 + z_2)$. In summary, Pedersen commitments over a multiplicative group \mathbb{G} are additively homomorphic, with addition of messages corresponds to group-multiplication of commitments.

Perfect HVZK Proof of Knowledge of Opening. We will see that in the design of general-purpose zero-knowledge arguments, it will occasionally be useful for the prover to prove that it knows how to open a commitment c to some value, *without* actually opening the commitment. As observed by Schnorr, Pedersen commitments have this property, using similar techniques to his Σ -protocol for the Discrete Logarithm relation. See Protocol 7.

The idea is that, for \mathcal{P} to prove it knows m, z such that $c = g^m h^z$, in the first round of the proof, the prover sends a group element $a \leftarrow g^d \cdot h^r$ for a random pair of exponents d, r . One should think of a as $\text{Com}(d, r)$, i.e., a commitment to d using randomness r . Then the verifier sends a random challenge e , and the verifier on its own can derive a commitment to $me + d$ via additive homomorphism, and the prover can derive an opening for this commitment. Specifically, $g^{me+d} \cdot h^{ze+r}$ commits to $me + d$, using randomness $ze + r$. Finally, the prover responds with opening information $(me + d, ze + r)$ for this derived commitment. An equivalent description of the protocol using this perspective is given in Protocol 8.

The idea for why the protocol is zero-knowledge is that since the verifier never learns d or r , the quantities $me + d$ and $ze + r$ that the prover sends to the verifier simply appear to be random elements modulo $|\mathbb{G}|$ from the verifier’s perspective. The intuition for why this is special sound is that since the committer does not know e before choosing d , there is no way for the prover to open the commitment to $me + d$ unless it knows how to open the commitment to m . In more detail, if the input commitment is $\text{Com}(m, z) = g^m h^z$, and \mathcal{P} ’s first message in the protocol is $a = g^d h^r$, then if \mathcal{P} can produce two accepting transcripts $(a, e, (m', z'))$ and $(a, e', (m'', z''))$ with $e \neq e'$, \mathcal{V} ’s acceptance criterion roughly implies that $m' = m \cdot e + d$ and $z' = z \cdot e + r$ while $m'' = m \cdot e' + d$ and $z'' = z \cdot e' + r$. These are four linearly independent equations in four unknowns, namely m, z, d , and r . Hence, one can take these two transcripts and efficiently solve for both m and z , as $m = (m' - m'') / (e - e')$ and $z = (z' - z'') / (e - e')$.

These intuitions are made formal below.

Protocol 7 Zero-Knowledge Proof of Knowledge of Opening of Pedersen Commitment

- 1: Let \mathbb{G} be a (multiplicative) cyclic group of prime order over which the Discrete Logarithm relation is hard, with randomly chosen generators g and h .
 - 2: Input is $c = g^m \cdot h^z$. Prover knows m and z , Verifier only knows c, g, h .
 - 3: Prover picks $d, r \in \{0, \dots, |\mathbb{G}| - 1\}$ and sends to verifier $a \leftarrow g^d \cdot h^r$.
 - 4: Verifier sends challenge e chosen at random from $\{0, \dots, |\mathbb{G}| - 1\}$.
 - 5: Prover sends $m' \leftarrow me + d$ and $z' \leftarrow ze + r$.
 - 6: Verifier checks that $g^{m'} \cdot h^{z'} = c^e \cdot a$.
-

Protocol 8 Equivalent Exposition of Protocol 7 in terms of commitments and additive homomorphism.

- 1: Let \mathbb{G} be a (multiplicative) cyclic group of prime order over which the Discrete Logarithm relation is hard, with randomly chosen generators g and h .
 - 2: Let $\text{Com}(m, z)$ denote the Pedersen commitment $g^m \cdot h^z$. Prover knows m and z , Verifier only knows $\text{Com}(m, z), g, h$.
 - 3: Prover picks $d, r \in \{0, \dots, |\mathbb{G}| - 1\}$ and sends to verifier $a \leftarrow \text{Com}(d, r)$.
 - 4: Verifier sends challenge e chosen at random from $\{0, \dots, |\mathbb{G}| - 1\}$.
 - 5: Let $m' \leftarrow me + d$ and $z' \leftarrow ze + r$, and let $c' \leftarrow \text{Com}(m', z')$. While Verifier does not know m' and z' , Verifier can derive c' unaided from $\text{Com}(m, z)$ and $\text{Com}(d, r)$ using additive homomorphism.
 - 6: Prover sends (m', z') .
 - 7: Verifier checks that m', z' is valid opening information for c' , i.e., that $g^{m'} \cdot h^{z'} = c'$.
-

Perfect Completeness. If prover follows the prescribed protocol in Protocol 7 then

$$g^{m'} \cdot h^{z'} = g^{me+r} \cdot h^{ze+r} = c^e \cdot a.$$

Special Soundness. Given two accepting transcripts $(a, e, (m'_1, z'_1))$ and $(a, e', (m'_2, z'_2))$ with $e \neq e'$, we have to extract a valid opening (m, z) for the commitment c , i.e., $g^m \cdot h^z = c$. As in the analysis of the Σ -protocol for the Discrete Logarithm relation, let $(e - e')^{-1}$ denote the multiplicative inverse of $e - e'$ modulo $|\mathbb{G}|$, and define

$$\begin{aligned} m^* &= (m'_1 - m'_2) \cdot (e - e')^{-1} \pmod{|\mathbb{G}|}, \\ z^* &= (z'_1 - z'_2) \cdot (e - e')^{-1} \pmod{|\mathbb{G}|}. \end{aligned}$$

Then

$$g^{m^*} \cdot h^{z^*} = \left(g^{(m'_1 - m'_2)} h^{(z'_1 - z'_2)} \right)^{(e - e')^{-1}} = \left(c^e \cdot a \cdot (c^{e'} \cdot a)^{-1} \right)^{(e - e')^{-1}} = c,$$

where the penultimate equality follows from the fact that $(a, e, (m'_1, z'_1))$ and $(a, e', (m'_2, z'_2))$ are accepting transcripts. That is, (m^*, z^*) is a valid (message, opening information) pair for the commitment c .

Perfect HVZK. The simulator samples e, m', z' uniformly at random from $\{0, \dots, |\mathbb{G}| - 1\}$ and then sets

$$a \leftarrow g^{m'} \cdot h^{e'} \cdot c^{-e},$$

and outputs

$$(a, e, (m', z')).$$

This ensures that e is uniformly distributed, and a , and (m', z') are also uniformly distributed over \mathbb{G} and $\{0, \dots, |\mathbb{G}| - 1\}^2$ under the constraint that $g^{m'} \cdot h^{e'} = c^e \cdot a$. This is the same distribution as that generated by the honest verifier interacting with the prescribed prover.

Perfect HVZK Proof of Knowledge of Opening to A Specific Value. The above protocol allows the prover to establish it knows how to open a Pedersen commitment c to *some* value. A variant we will also find useful allows the prover to establish in zero-knowledge that it knows how to open c to a specific public value y . Since a Pedersen commitment c to public value y is of the form $g^y h^r$ for a random $r \in \mathbb{G}$, proving that knowledge of how to open c to y is equivalent to proving knowledge of a value r such that $h^r = c \cdot g^{-y}$. This amounts to proving knowledge of the discrete logarithm of $c \cdot g^{-y}$ in base h , which can be done using Protocol 3.

A Final Perspective on Protocol 7. Protocol 7 asks the prover *not* to open c itself (which would violate zero-knowledge), but instead to open a different commitment c' , to random group element that is derived homomorphically from both c and a commitment to random value d that the prover sends via its first message. Both the prover and verifier “contribute randomness” to the value $m' = me + d$ committed by c' . The randomness contributed by the prover (namely d) is used to ensure that m' is statistically independent of m , which ensures that the opening m' for c' reveals no information about m . The verifier’s contribution e to m' is used to ensure special soundness: the prover cannot open c' for more than one value of the verifier’s challenge e unless the prover knows how to open c .

We will see more twists on this paradigm (in Sections 11.3.2 and 13.1.2), in contexts where the prover wants to establish in zero-knowledge that various committed values satisfy certain relationships. Directly opening the commitments would enable the verifier to easily check the claimed relationship, but violate zero-knowledge. So instead the prover opens *derived* commitments, with both the prover and verifier contributing randomness to the derived commitments in a manner such that the derived commitments satisfy the same property that the prover claims is satisfied by the original commitments.

11.3.2 Establishing A Product Relationship Between Committed Values

We have already seen the Pedersen commitments are additively homomorphic, meaning the verifier can take two commitments c_1 and c_2 to values m_1 and m_2 in $\{0, \dots, |\mathbb{G}| - 1\}$, and without any help from the committer, the verifier can derive a commitment to $m_1 + m_2$ (despite the fact that the verifier has no idea what m_1 and m_2 are, owing to the hiding property of the commitments).

Unfortunately, Pedersen commitments are not *multiplicatively* homomorphic: there is no way for the verifier to derive a commitment to $m_1 \cdot m_2$ without help from the committer. But suppose the committer *sends* a commitment c_3 that is claimed to be a commitment to value $m_1 \cdot m_2$ (meaning that the prover knows how to open up c_3 to the value $m_1 \cdot m_2$). Is it possible for the prover to prove to the verifier that c_3 indeed commits to $m_1 \cdot m_2$, without actually opening up c_3 and thereby revealing $m_1 \cdot m_2$? The answer is yes, using a somewhat more complicated variant of the Σ -protocols we have already seen. The Σ -protocol is depicted in Protocol 9, with an equivalent formulation in terms of commitments and additive homomorphism given in Protocol 10.

The rough idea of the protocol is that if m_3 indeed equals $m_1 \cdot m_2$, then c_3 can be thought of not only as a Pedersen commitment to $m_1 \cdot m_2$ using group generators g and h , i.e., $c_3 = \text{Com}_{g,h}(m_1 \cdot m_2, r_3)$, but also as a Pedersen commitment to m_2 using group generators $c_1 = g^{m_1} h^{r_1}$ and h . That is, if $m_3 = m_1 \cdot m_2$, it can be checked that

$$c_3 = \text{Com}_{c_1,h}(m_2, r_3 - r_1 m_2).$$

Equivalently, c_3 is a commitment to the same message m_2 as c_2 , just using a different generator (c_1 in place of g) and a different blinding factor ($r_3 - r_1 m_2$ in place of r_2). The protocol roughly enables the prover to establish in zero-knowledge that it knows how to open c_3 as a commitment of this form.

Similar to Protocol 7, the idea is to have the prover send commitments to random values b_1 and b_3 , the latter being committed twice, once using generators (g, h) and once using generators (c_1, h) . The verifier then derives commitments to $em_1 + b_1$ and $em_2 + b_3$ using additive homomorphism (with two commitments derived for the latter quantity, one under the pair of generators (g, h) and the other under (c_1, h)), and then the prover opens these derived commitments. Roughly speaking, the protocol is zero-knowledge since the random choice of b_1 and b_3 ensures that the revealed opening is a random group element independent of m_1 and m_2 .

In more detail, the prover first sends three values α, β, γ , where $\alpha = \text{Com}_{g,h}(b_1, b_2)$ and $\beta = \text{Com}_{g,h}(b_3, b_4)$ are commitments to random values $b_1, b_3 \in \{0, \dots, |\mathbb{G}| - 1\}$ using random blinding factors $b_2, b_4 \in \{0, \dots, |\mathbb{G}| - 1\}$. Here, the group generators used to produce the two commitments are g and h . γ on the other hand is *another* commitment to b_3 (just as β is), but using group generators c_1 and h rather than g and h . That is, γ is set to $\text{Com}_{c_1,h}(b_3, b_5)$ for a randomly chosen b_5 .

From these three values, and despite not knowing m_1, m_2, r_1, r_2, r_3 , or b_1, \dots, b_5 , the verifier can, for any value $e \in \mathbb{G}$, use additive homomorphism to derive commitments $c'_1 = \text{Com}_{g,h}(b_1 + em_1, b_2 + er_1)$, $c'_2 = \text{Com}_{g,h}(b_3 + em_2, b_4 + er_2)$, and $c'_3 = \text{Com}_{c_1,h}(b_3 + em_2, b_5 + e(r_3 - r_1m_2))$. After the verifier sends a random challenge e , the prover responds with five values z_1, \dots, z_5 such that (z_1, z_2) , (z_3, z_4) and (z_3, z_5) are opening information for c'_1, c'_2 and c'_3 respectively.

Completeness, special soundness, and honest-verifier zero-knowledge. Completeness holds by design. For brevity, we merely sketch the intuition for why special soundness and zero-knowledge hold (though the formal proofs are not difficult and can be found in [Mau09] or [WTS⁺18, Appendix A]).

The intuition for why the protocol is honest-verifier zero-knowledge is that the blinding factors b_2, b_4, b_5 ensure that the prover's first message (α, β, γ) leaks no information about the random committed values b_1, b_3 , and this in turn ensures that the prover's second message (z_1, \dots, z_5) reveal no information about m_1 and m_2 .

The intuition for why the protocol is special-sound is that if (a, e, z) and (a, e', z') are two accepting transcripts, where $a = (\alpha, \beta, \gamma)$, $z = (z_1, \dots, z_5)$, and $z' = (z'_1, \dots, z'_5)$ then the verifier's checks roughly ensure that:

- $b_1 + em_1 = z_1$ and $b_1 + e'm_1 = z'_1$.
- $b_2 + er_1 = z_2$ and $b_2 + e'r_1 = z'_2$.
- $b_3 + em_2 = z_3$ and $b_3 + e'm_2 = z'_3$.
- $b_4 + er_2 = z_4$ and $b_4 + e'r_2 = z'_4$.
- $b_5 + e(r_3 - r_1m_2) = z_5$ and $b_5 + e'(r_3 - r_1m_2) = z'_5$

The first two bullet points refer to the fact that (z_1, z_2) opens $\text{Com}_{g,h}(b_1 + em_1, b_2 + er_1)$ and (z'_1, z'_2) open $\text{Com}_{g,h}(b_1 + e'm_1, b_2 + e'r_1)$. As in the special soundness analysis of Protocol 7, if $e \neq e'$ then the first bullet point represents two linearly independent equations in the unknown m_1 and hence enables solving for m_1 as $(z_1 - z'_1) \cdot (e - e')^{-1}$. Similarly, the second bullet point enables solving for r_1 as $(z_2 - z'_2) \cdot (e - e')^{-1}$. Formally, one can show that $((z_1 - z'_1) \cdot (e - e')^{-1}, (z_2 - z'_2) \cdot (e - e')^{-1})$ is a valid opening for c_1 using generators g and h .

The next two bullet points refer to the fact that (z_3, z_4) opens $\text{Com}_{g,h}(b_3 + em_2, b_4 + er_2)$ and (z'_3, z'_4) open $\text{Com}_{g,h}(b_3 + e'm_2, b_4 + e'r_2)$, and enable solving for m_2 and r_2 as $(z_3 - z'_3) \cdot (e - e')^{-1}$ and $(z_4 - z'_4) \cdot (e - e')^{-1}$.

Formally, one can show that $((z_3 - z'_3) \cdot (e - e')^{-1}, (z_4 - z'_4) \cdot (e - e')^{-1})$ is a valid opening for c_2 using generators g and h .

The final bullet point refers to the fact that (z_3, z_5) opens $\text{Com}_{c_1, h}(b_3 + em_2, b_5 + e(r_3 - r_1m_2))$ and (z'_3, z'_5) opens $\text{Com}_{c_1, h}(b_3 + e'm_2, b_5 + e'(r_3 - r_1m_2))$. Since r_1 and m_2 have already been derived from the preceding bullet points, the two equations in the final bullet point enable solving for r_3 as $(z_5 - z'_5) \cdot (e - e')^{-1} + r_1m_2$. Formally, one can show that $(m_1 \cdot m_2, (z_5 - z'_5) \cdot (e - e')^{-1} + r_1m_2)$ is a valid opening for c_3 using generators g and h .

Protocol 9 Zero-Knowledge PoK of Opening of Pedersen Commitments Satisfying Product Relationship

- 1: Let \mathbb{G} be a (multiplicative) cyclic group of prime order over which the Discrete Logarithm relation is hard.
- 2: Input is $c_i = g^{m_i} \cdot h^{r_i}$ for $i \in \{1, 2, 3\}$ such that $m_3 = m_1 \cdot m_2 \pmod{|\mathbb{G}|}$.
- 3: Prover knows m_i and r_i for all $i \in \{1, 2, 3\}$, Verifier only knows c_1, c_2, c_3, g, h .
- 4: Prover picks $b_1, \dots, b_5 \in \{0, \dots, |\mathbb{G}| - 1\}$ and sends to verifier three values:

$$\alpha \leftarrow g^{b_1} \cdot h^{b_2}, \beta \leftarrow g^{b_3} \cdot h^{b_4}, \gamma \leftarrow c_1^{b_3} \cdot h^{b_5}.$$

- 5: Verifier sends challenge e chosen at random from $\{0, \dots, |\mathbb{G}| - 1\}$.
- 6: Prover sends $z_1 \leftarrow b_1 + e \cdot m_1, z_2 \leftarrow b_2 + e \cdot r_1, z_3 \leftarrow b_3 + e \cdot m_2, z_4 \leftarrow b_4 + e \cdot r_2, z_5 \leftarrow b_5 + e \cdot (r_3 - r_1m_2)$.
- 7: Verifier checks that the following three equalities hold:

$$g^{z_1} \cdot h^{z_2} = \alpha \cdot c_1^e,$$

$$g^{z_3} \cdot h^{z_4} = \beta \cdot c_2^e,$$

and

$$c_1^{z_3} \cdot h^{z_5} = \gamma \cdot c_3^e.$$

Protocol 10 Equivalent description of Protocol 9 in terms of commitments and additive homomorphism. The notation $\text{Com}_{g,h}(m, z) := g^m h^z$ indicates that the group generators used to produce the Pedersen commitment to m with blinding factor z are g and h .

- 1: Let \mathbb{G} be a (multiplicative) cyclic group of prime order over which the Discrete Logarithm relation is hard.
- 2: Input is $c_i = g^{m_i} \cdot h^{r_i} = \text{Com}_{g,h}(m_i, r_i)$ for $i \in \{1, 2, 3\}$ such that $m_3 = m_1 \cdot m_2 \pmod{|\mathbb{G}|}$.
- 3: Prover knows m_i and r_i for all $i \in \{1, 2, 3\}$, Verifier only knows c_1, c_2, c_3, g, h .
- 4: Prover picks $b_1, \dots, b_5 \in \{0, \dots, |\mathbb{G}| - 1\}$ and sends to verifier three values:

$$\alpha \leftarrow \text{Com}_{g,h}(b_1, b_2), \beta \leftarrow \text{Com}_{g,h}(b_3, b_4), \gamma \leftarrow \text{Com}_{c_1,h}(b_3, b_5).$$

- 5: Verifier sends challenge e chosen at random from $\{0, \dots, |\mathbb{G}| - 1\}$.
- 6: Let $z_1 \leftarrow b_1 + e \cdot m_1, z_2 \leftarrow b_2 + e \cdot r_1, z_3 \leftarrow b_3 + e \cdot m_2, z_4 \leftarrow b_4 + e \cdot r_2, z_5 \leftarrow b_5 + e \cdot (r_3 - r_1 m_2)$.
- 7: While Verifier does not know z_1, \dots, z_5 , using additive homomorphism Verifier can derive the following three commitments unaided using additive homomorphism:

$$c'_1 = \text{Com}_{g,h}(z_1, z_2) = \alpha \cdot c_1^e,$$

$$c'_2 = \text{Com}_{g,h}(z_3, z_4) = \beta \cdot c_2^e,$$

$$c'_3 = \text{Com}_{c_1,h}(z_3, z_5) = \gamma \cdot c_3^e.$$

This final equality for c'_3 exploits that

$$c_3^e = g^{em_1 m_2} h^{er_3} = c_1^{em_2} h^{er_3 - er_1 m_2} = \text{Com}_{c_1,h}(em_2, er_3 - er_1 m_2).$$

- 8: Prover sends z_1, \dots, z_5 .
- 9: Verifier checks that:

- (z_1, z_2) is valid opening information for c'_1 using generators g, h .
 - (z_3, z_4) is valid opening information for c'_2 using generators g, h .
 - (z_3, z_5) is valid opening information for c'_3 using generators c_1, h .
-

Chapter 12

Zero-Knowledge via Commit-And-Prove and Masking Polynomials

Historically, the first zero-knowledge argument for an **NP**-complete problem was given by Goldreich, Micali, and Wigderson (GMW) [GMW91]. GMW designed a zero-knowledge argument with a polynomial-time verifier for the Graph 3-Coloring problem. This yields a zero-knowledge argument with a polynomial time verifier for any language \mathcal{L} in **NP** (including arithmetic circuit satisfiability), because any instance of \mathcal{L} can first be transformed into an equivalent instance of Graph 3-Coloring with a polynomial blowup in instance size, and then GMW’s zero-knowledge argument for Graph 3-Coloring can be applied. However, this does not yield a practical protocol for two reasons. First, GMW’s construction works by first designing a “basic” protocol that has large soundness error $(1 - 1/|E|)$, where $|E|$ denotes the number of edges in the graph) and hence needs to be repeated a polynomial number of times to ensure negligible soundness error. Second, for problems in **NP** that are relevant in practice, reductions to Graph 3-Coloring can introduce large (polynomial) overheads. That is, we saw in Chapter 5 that arbitrary non-deterministic RAMs running in time T can be transformed into equivalent circuit satisfiability instances of size $\tilde{O}(T)$, but an analogous result is not known for Graph 3-Coloring. For this reason, our focus in this manuscript is on directly giving zero-knowledge arguments for circuit satisfiability and related problems, rather than for other **NP**-complete problems. The interested reader can learn more about GMW’s seminal zero-knowledge argument from any standard text on zero-knowledge (e.g., [Gol07, Section 4.4.2]).

Commit-and-prove zero-knowledge arguments. In this chapter, we describe our first zero-knowledge arguments for circuit satisfiability. These are based on a technique often called *commit-and-prove*.¹²⁴ The idea is as follows. Suppose that for some agreed-upon circuit \mathcal{C} , the prover wants to establish that it knows a witness w such that $\mathcal{C}(w) = 1$,¹²⁵ and consider the following naive, information-theoretically secure and non-interactive proof system, which is (perfectly) sound but not zero-knowledge. The prover sends w to the

¹²⁴An important warning: some papers use the phrase “commit-and-prove SNARKs”, e.g., [CFQ19], which is related to but different than our use of the term commit-and-prove in this survey. Commit-and-prove SNARKs are SNARKs in which the verifier is given a compressing commitment to an input vector (e.g., using the generalized Pedersen commitment we describe later in Section 13.1.2), and the SNARK is capable of establishing that the prover knows an opening w for the commitment such that w satisfies a property of interest. Hence, commit-and-prove SNARKs are SNARKs for a particular type of statement. In contrast, we use commit-and-prove to refer to a particular *design approach* for zero-knowledge arguments.

¹²⁵In previous chapters of this survey, we have considered arithmetic circuits that take as input a public input x and witness w , and the prover wants to establish knowledge of a w such that $\mathcal{C}(x, w) = 1$. In this chapter we omit the public input x for brevity. It is easy to modify the arguments given here to support a public input x in addition to a witness w .

verifier, along with the value of every gate of \mathcal{C} when \mathcal{C} is evaluated on input w . The verifier simply checks that the claimed value of the output gate is 1, and checks gate-by-gate that the claimed value of the gate is accurate (i.e., for any multiplication (respectively, addition) gate, the value the prover sends for that gate is indeed the product (respectively, sum) of the two in-neighbors of the gate). Clearly, this proof system is information-theoretically sound, but is not zero-knowledge because the verifier learns the witness w .

To obtain a zero-knowledge argument, the prover will instead send a hiding commitment to each gate, and prove in zero-knowledge that the *committed* values satisfy the checks that the verifier in the naive (non-zero-knowledge) proof system performs. This way the argument system verifier learns nothing about the committed values, but nonetheless confirms that the committed values would have satisfied the verifier within the information-theoretically secure protocol.

The next section contains additional details of this approach when the commitment scheme used is Pedersen commitments.

12.1 Proof Length of Witness Size Plus Multiplicative Complexity

Section 11.3.2 explained that Pedersen commitments satisfy the following properties: (a) they are additively homomorphic, meaning that given commitments c_1, c_2 to values m_1, m_2 , the verifier can compute a commitment to $m_1 + m_2 \pmod{|\mathbb{G}|}$ directly from c_1, c_2 , even though the verifier does not know m_1 or m_2 (b) given commitments c_1, c_2, c_3 to values m_1, m_2, m_3 there is a Σ -protocol (Protocol 9) for which the prover can establish in (honest verifier) zero-knowledge that c_3 is a commitment to $m_1 \cdot m_2 \pmod{|\mathbb{G}|}$.

Addition and multiplication are a universal basis, meaning that with these two operations alone, one can compute *arbitrary* functions of any input. Hence, properties (a) and (b) together effectively mean that a verifier is able to do arbitrary computation over committed values, without making the prover ever reveal the committed values.

In more detail, we have the following zero-knowledge argument for arithmetic circuit satisfiability. While conceptually appealing, this argument is not succinct—the communication complexity is linear in the witness size $|w|$ plus the number M of multiplication gates of the circuit, leading to very large proofs.

Let \mathcal{C} be an arithmetic circuit over \mathbb{F} of prime order, and let \mathbb{G} be a cyclic group of the same order as \mathbb{F} in which the Discrete Logarithm relation is assumed to be hard. Let us suppose that multiplication gates in \mathcal{C} have fan-in 2 (the zero-knowledge argument in this section naturally supports addition gates of unbounded fan-in, in which case we can assume without loss of generality that the in-neighbors of any addition gate consist entirely of multiplication gates). Suppose the prover claims that it knows a w such that $\mathcal{C}(w) = 1$.

At the start of the protocol, the prover sends Pedersen commitments to each entry of w , as well as Pedersen commitments to the value of every multiplication gate in \mathcal{C} . Then, for each entry of witness w , the prover proves via Protocol 7 that the prover knows an opening of the commitment to that entry. Next, for each multiplication gate in the circuit, the prover uses Protocol 9 to prove that the committed values respect the operations of the multiplication gates. That is, if a multiplication gate g_1 computes the product of gates g_2 and g_3 , the verifier can demand that the prover prove in zero-knowledge that the commitment c_1 to the value of gate g_1 equals the product of the commitments c_2 and c_3 to the value of gates g_2 and g_3 . Addition gates are handled within the protocol without any communication between prover and verifier by using the additive homomorphism property of Pedersen commitments: if an addition gate g_1 computes the sum of gates g_2 and g_3 , the verifier can on its own, via Property (a), compute a commitment to the value of g_1 given commitments to the values of gates g_2 and g_3 . Finally, at the end of the protocol, the prover uses Protocol 3 to prove knowledge of how to open the commitment to the value of the output gate of \mathcal{C} to value $y = 1$.

The resulting proof system is clearly complete because each of the subroutines (Protocols 3, 7, and 9)

is complete. To show it is perfect honest-verifier zero-knowledge, one must construct an efficient simulator whose output is distributed identically to the honest verifier’s view in the protocol. The idea of the construction is simply that the protocol is comprised entirely of $|w| + M + 1$ sequential invocations of Σ -protocols that are themselves perfect honest-verifier zero-knowledge. The simulator for the entire protocol can simply run the simulator for each of these subroutines in sequence and concatenate the transcripts that it generates.

12.1.1 Establishing Knowledge Soundness

To establish our argument system is an argument of knowledge for arithmetic circuit satisfiability, we need to show that if the prover convinces the verifier to accept with non-negligible probability, then it is possible to efficiently extract from the prover a witness w such that $\mathcal{C}(w) = 1$. Formally, we must show that for any prover \mathcal{P} that convinces the argument system verifier to accept with non-negligible probability, there is a polynomial-time algorithm \mathcal{E} that, given access to a rewindable transcript generator for the prover-verifier pair $(\mathcal{P}, \mathcal{V})$ for the argument system, outputs a w such that $\mathcal{C}(w) = 1$.

Naturally, the procedure to extract the witness w relies on the fact that each of the $|w| + M + 1$ subroutines used in the argument system protocol themselves satisfies special soundness. Recall that this means the protocols consist of three messages, and given access to two accepting transcripts that share a first message and differ in their second message, there is an efficient procedure to extract a witness for the statement being proven. We call such a set of transcripts a 2-transcript-tree for the subroutine.

Using its access to a rewindable transcript generator for $(\mathcal{P}, \mathcal{V})$, \mathcal{E} can in polynomial time identify a 2-transcript-tree for each subroutine with high probability.¹²⁶

By special soundness of Protocol 7, given such a set of 2-transcript-trees for all of the subroutines of the argument system, \mathcal{E} can extract an opening of the commitment to each entry i of the witness, and output the vector w of extracted values. We now explain that the vector w that is output in this manner indeed satisfies $\mathcal{C}(w) = 1$.

Just as \mathcal{E} extracted an opening for each entry of w from the 2-transcript-trees for each invocation of Protocol 7, given the 2-transcript-trees for the invocation of Protocol 9 to the i th multiplication gate of \mathcal{C} , there is an efficient procedure to extract openings to the commitment for multiplication gate i and the commitments to its two in-neighbor gates such that the values opened respect the multiplication operation (one or both of these in-neighbors may be addition gates, the commitments for which are derived via additive homomorphism from the commitments to multiplication gates sent by the prover). Similarly, given the 2-transcript-tree for the lone invocation of Protocol 3, there is an efficient procedure to extract an opening of the commitment to the output gate value to 1.

Observe that a value for any particular gate g in \mathcal{C} may be extracted multiple times by these extraction procedures. For example, the value of a gate g will be extracted via the 2-transcript-tree for any invocation of Protocol 9 to a gate g' for which g is an in-neighbor. And if g is itself a multiplication gate, its value will

¹²⁶The *forking lemma* covered in Section 11.2.3, showed how to generate a 2-transcript-tree given a convincing prover for any Σ -protocol, even one that has been rendered non-interactive via the Fiat-Shamir transformation. If the Fiat-Shamir transformation has not been applied, generating a 2-transcript-tree for a Σ -protocol is even simpler: run $(\mathcal{P}, \mathcal{V})$ once to generate an accepting transcript for the Σ -protocol, then rewind the Σ -protocol to just after \mathcal{P} sent its first message, and restart the Σ -protocol with a fresh random challenge to generate a new transcript. One can show that both generated transcripts will be accepting with probability at least $\Omega(\epsilon^2)$ where ϵ is the probability \mathcal{P} passes \mathcal{V} ’s checks in the Σ -protocol. And with overwhelming probability the two verifier challenges in the two transcripts will be distinct (specifically, with probability at least $1 - 1/2^\lambda$, where 2^λ is the size of the set from which the verifier’s challenge is chosen). In this event, the two transcripts form a 2-transcript tree. This procedure can be repeated $O(\log |\mathcal{C}|/\epsilon^2)$ times for each subroutine in the argument system applied to the arithmetic circuit \mathcal{C} , to ensure that the probability of successfully generating at least one 2-transcript-tree for every subroutine is at least, say, 9/10. See Theorem 13.1 in Section 13.1.4 for a substantial generalization of this procedure for generating transcript trees, to multi-round protocols.

be extracted an additional time from the application of Protocol 9 to g itself. And the output gate of \mathcal{C} will have an opening of its commitment to value 1 extracted due to the invocation of Protocol 3.

For any gate whose value is extracted multiple times, the extracted values must all be the same, for if this were not the case, the extraction procedure would have identified two different openings of the same commitment. This would violate the binding property of the commitment scheme, since all the 2-transcript-trees were constructed in polynomial time and the extraction procedure from each 2-transcript-tree is also efficient.

In summary, we have established the following properties of the extracted values:

- A unique value is extracted for every gate of \mathcal{C} and entry of the witness w .
- The extracted values for all multiplication gates respect the multiplication operation of the gate (this holds by the special soundness of Protocol 9).
- The extracted values of the gates also respect the addition operations computed by all addition gates of \mathcal{C} (this holds by the additive homomorphism of the commitment scheme).
- The extracted value for the output gate is 1.

This four properties together imply that $\mathcal{C}(w) = 1$ where w is the extracted witness.

12.1.2 Commit-and-Prove with Other Commitment Schemes

We used Pedersen commitments in the commit-and-prove zero-knowledge argument system above. However, the only properties of Pedersen commitments we needed were: perfect hiding, computational binding, additive homomorphism, and zero-knowledge arguments of knowledge for opening information and product relationships. One can replace Pedersen commitments with any other commitment scheme satisfying these properties. To this end, several works [WYKW20, BMRS20a, DIO20] essentially replace Pedersen commitments with a commitment scheme derived from a primitive called *vector oblivious linear evaluation* (VOLE) [ADI⁺17]. This has the following benefits over Pedersen commitments. First, using Pedersen commitments to implement commit-and-prove leads to proofs containing 10 elements of a cryptographic group per multiplication gate. The use of VOLE-based commitments can reduce this communication to as low as 1 or 2 field elements per multiplication gate. Second, the computational binding property of Pedersen commitments is based on the intractability of the discrete logarithm problem, and since quantum computers can efficiently compute discrete logarithms, the resulting commit-and-prove arguments are not quantum-sound. In contrast, VOLE-based commitments are believed to be quantum-sound (they are based on variants of the so-called Learning Parity with Noise (LPN) assumption).

However, the use of VOLE-based commitments comes with significant downsides as well. Specifically, these commitments currently require an interactive pre-processing phase. Unlike commit-and-prove with Pedersen-based commitments, the interaction cannot be fully removed with the Fiat-Shamir transformation, and accordingly the resulting arguments for circuit satisfiability are not publicly verifiable.

12.2 Avoiding Linear Dependence on Multiplicative Complexity: zk-Arguments from IPs

The proof length in the zero-knowledge argument of the previous section is linear in the witness length and number of multiplication gates of \mathcal{C} . Moreover, the verifier's runtime is linear in the size of \mathcal{C} (witness length

plus number of addition and multiplication gates), as the verifier effectively applies every gate operation in \mathcal{C} “underneath the commitments” (i.e., the verifier evaluates every gate on committed values, without ever asking the prover to open any commitments).

It is possible to reduce the communication complexity and verifier runtime to $O(|w| + d \cdot \text{polylog}(|\mathcal{C}|))$, where d is the depth of $|\mathcal{C}|$, by combining the ideas of the previous section with the GKR protocol. The idea is to start with our first, naive protocol for circuit satisfiability, (Section 6.1), which is not zero-knowledge, and combine it with the ideas of the previous section to render it zero-knowledge without substantially increasing any of its costs (communication, prover runtime, or verifier runtime). Specifically, recall that in the naive protocol of Section 6.1 we had the prover explicitly send w to the verifier, and then applied the GKR protocol to prove that $\mathcal{C}(w) = 1$. This was not zero-knowledge because the verifier learns the witness w .

To render it zero-knowledge, we can have the prover send Pedersen commitments to each element of w and use Protocol 7 to prove knowledge of openings of each commitment, exactly as the prover did at the start of the zero-knowledge argument from the previous section. Then we can apply the GKR protocol to the claim that $\mathcal{C}(w) = 1$. However, the prover’s messages within the GKR protocol also leak information about the witness w to the verifier, as the prover’s messages all consist of low-degree univariate polynomials whose coefficients are derived from \mathcal{C} ’s gate values when evaluated on w . To address this issue, we do not have the prover send the coefficients of these polynomials to the verifier “in the clear”, but rather have the prover \mathcal{P} send Pedersen commitments to these coefficients and engage for each one in an invocation of Protocol 7 to prove that \mathcal{P} knows an opening of the commitment. In sum, when the argument system prover and verifier have finished simulating the GKR protocol, the argument system prover has sent Pedersen commitments to all entries of the witness w and all entries of the GKR prover’s messages.

We now have to explain how the argument system verifier can confirm in zero-knowledge that the values inside these commitments would have convinced the GKR verifier to accept the claim that $\mathcal{C}(w) = 1$. The idea is roughly that there is a circuit \mathcal{C}' that takes as input the prover’s messages in the GKR protocol (including the witness w), and such that (1) all of \mathcal{C}' outputs are 1 if and only if the prover’s messages would convince the GKR verifier to accept, and (2) \mathcal{C}' contains $O(d \log |\mathcal{C}| + |w|)$ addition and multiplication gates. Hence, we can apply the zero-knowledge argument of the previous section to the claim that \mathcal{C}' outputs the all-1s vector. Recall that at the start of the argument system of the previous section applied to the claim $\mathcal{C}'(w') = 1$, the prover sent commitments to each entry of w' . In this case, w' consists of the witness w for \mathcal{C} and the prover’s messages within the GKR protocol, and the argument system has already committed to these values (via the last sentence of the previous paragraph). By Property (2) of \mathcal{C}' , the total communication cost and verifier runtime of the zero-knowledge argument applied to \mathcal{C}' is $O(|w| + d \log |\mathcal{C}|)$.

The argument for \mathcal{C} is easily seen to be complete and honest-verifier zero-knowledge (since it consists of the sequential application of honest-verifier zero-knowledge argument systems). To formally prove that it is knowledge sound, one needs to show that, given any argument system prover \mathcal{P} that runs in polynomial time and causes the argument system verifier to accept with non-negligible probability, one can extract a witness w and a prover strategy \mathcal{P}' for the GKR protocol applied to the claim $\mathcal{C}(w) = 1$ that causes the GKR verifier to accept with high probability. Soundness of the GKR protocol then implies that $\mathcal{C}(w) = 1$. The witness w can be extracted from \mathcal{P} as in the previous section via the special soundness of Protocol 7. In each round of the GKR protocol, the message to be sent by the GKR prover \mathcal{P}' in response to the GKR verifier’s challenge can also be extracted from the commitments sent by the argument system prover \mathcal{P} in response to the same challenge, because Protocol 7 was invoked in each round of the argument system to prove that \mathcal{P}' knows openings to every commitment sent.

The probability that the GKR verifier accepts when interacting with \mathcal{P}' is analyzed by exploiting the fact

that the GKR verifier’s checks on the committed messages sent by \mathcal{P}' are performed by applying the zero-knowledge proof of knowledge of the previous section to \mathcal{C}' . Specifically, soundness of the argument system applied to \mathcal{C}' ensures that whenever \mathcal{P} convinces the argument system verifier to accept, \mathcal{P}' convinces the GKR verifier to accept (up to the negligible probability with which a polynomial time adversary is able to break binding property of the commitment scheme).

To summarize, in the above argument system, we essentially applied the commit-and-prove zero-knowledge argument of Section 12.1 not to \mathcal{C} itself, but rather to the verifier in the GKR protocol applied to check the claim that $\mathcal{C}(w) = 1$.

Reducing the dependence on witness size below linear. The argument system just described has communication complexity that grows linearly with $|w|$, because the prover sends a hiding commitment to each entry of w and proves in zero-knowledge that it knows openings to each commitment. The next chapter describes several practical polynomial commitment schemes. Rather than committing to each entry of w individually, the prover could commit to the multilinear extension \tilde{w} of w using an extractable polynomial commitment scheme as outlined in Section 6.3, and thereby reduce the dependence on the proof length on $|w|$ from linear to sublinear or even logarithmic. (More precisely, to ensure zero-knowledge, the polynomial commitment scheme should be hiding, and during its evaluation phase it should reveal to the verifier a hiding *commitment* to $\tilde{w}(z)$ for any point z chosen by the verifier. See for example the multilinear polynomial commitment scheme in Section 13.1.3.)

This same approach also transforms the succinct MIP-derived argument of Section 7.3 into a zero-knowledge one. Specifically, after having the argument system prover first commit to the multilinear polynomial Z claimed to extend a valid circuit transcript (with a hiding commitment scheme), the prover and verifier then simulate the MIP verifier’s interactions with the first MIP prover, but with the prover sending Pedersen commitments to the MIP prover’s messages rather than the messages themselves, and proves in zero-knowledge that it knows openings for the commitments. The argument system verifier then confirms in zero-knowledge that the values inside these commitments would have convinced the MIP verifier to accept the claim.

The ideas in this section and the previous section were introduced by Cramer and Damgård [CD98] and first implemented and rendered practical via optimizations in [WTS⁺18, ZGK⁺17b, Set19].

12.3 Zero-Knowledge via Masking Polynomials

The preceding section (Section 12.2) gave a generic technique for transforming any IP into a zero-knowledge argument: the argument system prover mimics the IP prover, but rather than sending the IP prover’s messages in the clear, it sends hiding commitments to those messages, and proves in zero-knowledge that it knows how to open the commitments. At the end of the protocol, the argument system prover establishes in zero-knowledge that the committed messages would have caused the IP verifier to accept; this is done efficiently by exploiting homomorphism properties of the commitments.

In this section, we discuss another technique for transforming any IP into a zero-knowledge argument. The technique makes use of any extractable polynomial commitment scheme, meaning that we assume the prover is able to cryptographically bind itself to a desired polynomial p , and later the verifier can force the prover to reveal the evaluation $p(r)$ for a random input r to p of the verifier’s choosing. Suppose further that the polynomial commitment scheme is zero-knowledge, meaning that the verifier learns nothing about p from the commitment, and the evaluation phase reveals no information about p to the verifier other than

the evaluation $p(r)$. One benefit of this technique is that if the polynomial commitment scheme is binding even against quantum cheating provers, then the resulting zero-knowledge argument is also plausibly post-quantum sound. For example, the FRI-based polynomial commitment scheme of Section 9.2.2 is plausibly sound against cheating provers that can run polynomial time quantum algorithms.¹²⁷ In contrast, any protocol (such as that of the last section) that makes use of Pedersen commitments is not post-quantum sound, because Pedersen commitments are only binding if the discrete logarithm problem is intractable, and quantum computers can compute discrete logarithms in polynomial time.

Another zero-knowledge sum-check protocol. Consider applying the sum-check protocol to an ℓ -variate polynomial g over \mathbb{F} to check the prover’s claim that $\sum_{x \in \{0,1\}^\ell} g(x)$ equals some value G . Let us assume that the verifier has oracle access to g , in the sense that for any point $r \in \mathbb{F}^\ell$, the verifier can obtain $g(r)$ with one query to the oracle. Recall (Section 4.1) that the sum-check protocol consists of ℓ rounds, where the honest prover’s message in each round i is a univariate polynomial of degree $\deg_i(g)$ derived from g , namely

$$\sum_{b_{i+1}, \dots, b_\ell \in \{0,1\}} g(r_1, \dots, r_{i-1}, X_i, b_{i+1}, \dots, b_\ell).$$

Here, $\deg_i(g)$ is the degree of g in variable i and is assumed known to the verifier, and r_1, \dots, r_{i-1} are random field elements chosen by the verifier in rounds $1, 2, \dots, i-1$.

There are three ways in which the verifier “learns” information about g in the sum-check protocol. First, the verifier learns that $\sum_{x \in \{0,1\}^\ell} g(x) = G$, but this information is not meant to be “protected” as the entire point of the sum-check protocol is to ensure that the verifier learns this value. Second, the prover’s messages leak information about g to the verifier that the verifier may not be able to compute on her own. Third, the verifier learns the value $g(r)$ via the oracle query at the end of the protocol.

In the preceding section, we addressed the second source of information leakage by having the prover send hiding commitments to the messages rather than the messages themselves. Here is a different technique for ensuring that the prover’s messages do not leak any information about g to the verifier; this approach originated in [BSCF⁺17, CFGS18].

To ensure that the prover’s messages in the sum-check protocol reveal no information about g , the prover can at the very start of the protocol choose a random polynomial p with the same degree as g in each variable, commit to p , and send to the verifier a value P claimed to equal $\sum_{x \in \{0,1\}^\ell} p(x)$. The verifier then picks a random $\rho \in \mathbb{F} \setminus \{0\}$ and sends it to the prover, and the prover and verifier apply the sum-check protocol not to g itself but rather to $g + \rho \cdot p$, to check that $\sum_{x \in \{0,1\}^\ell} (g + \rho \cdot p)(x) = G + \rho \cdot P$.

At the end of the sum-check protocol, the verifier needs to evaluate $g + \rho \cdot p$ at a random input $r \in \mathbb{F}^\ell$. The value $p(r)$ can be obtained via the evaluation phase of the commitment scheme that was applied to p , while $g(r)$ is obtained by the verifier with a single oracle query.

Completeness and soundness. The protocol clearly satisfies completeness. To see that it is sound, consider any prover strategy \mathcal{P} capable of convincing the verifier to accept with non-negligible probability. By extractability of the polynomial commitment scheme, it is possible to efficiently extract from \mathcal{P} a polynomial p such that the prover is bound to p , in the sense that any value revealed by the prover in the evaluation phase of the commitment scheme is consistent with p . Letting P be the claimed value of $\sum_{x \in \{0,1\}^\ell} p(x)$ sent by \mathcal{P} , consider the two functions $\pi_1(\rho) = G + \rho P$ and $\pi_2(\rho) = \sum_{x \in \{0,1\}^\ell} (g + \rho \cdot p)(x)$. Both are linear functions in ρ . If either $G \neq \sum_{x \in \{0,1\}^\ell} g(x)$ or $P \neq \sum_{x \in \{0,1\}^\ell} p(x)$ then $\pi_1 \neq \pi_2$, and hence the two linear functions

¹²⁷FRI-derived polynomial commitments are not zero-knowledge, but can be rendered zero-knowledge using techniques similar to those in this section.

can agree on at most one value of ρ . This means that with probability at least $1 - \frac{1}{|\mathbb{F}|-1}$ over the random choice of ρ , $G + \rho P \neq \sum_{x \in \{0,1\}^\ell} (g + \rho \cdot p)(x)$. In this event, the sum-check protocol is applied to a false claim, and we conclude that the verifier will reject with high probability because the sum-check protocol is sound.

Honest-Verifier Zero-Knowledge. We claim that the honest verifier in this protocol learns nothing about g other than G and $g(r)$. This is formalized by giving an efficient simulator that, given G and the ability to query g at a single input r , produces a distribution identical to that of the prover's messages in the above protocol.

The intuition is that, since p is random, adding $\rho \cdot p$ to g yields a random polynomial satisfying the same degree bounds as g , and hence the prover's messages in the sum-check protocol applied to $g + \rho \cdot p$ are indistinguishable from those obtained by applying the sum-check protocol to a randomly chosen polynomial. Formally, the simulator selects a random polynomial p subject to the appropriate degree bounds (i.e., $\deg_i(p) = \deg_i(g)$ for all i), commits to p exactly as does the honest prover in the protocol above (here, we are using the fact that p is chosen totally independent of g and hence the simulator can commit to p even though the simulator has no knowledge of g), and sets $P \leftarrow \sum_{x \in \{0,1\}^\ell} p(x)$. The simulator then chooses ρ at random from $\mathbb{F} \setminus \{0\}$, and chooses a random value $r = (r_1, \dots, r_\ell) \in \mathbb{F}^\ell$. The simulator queries the oracle for g at r to obtain $g(r)$ and then chooses a random polynomial f subject to the constraint that f sums to G over inputs in $\{0,1\}^\ell$ and $f(r) = g(r)$ (this can be done in time $O(2^\ell)$, which is polynomial in n if $\ell = O(\log n)$). The simulator then computes the honest prover's messages in the sum-check protocol applied to $f + \rho p$ when the sum-check verifier's randomness is r . At the end of the sum-check protocol, when the verifier needs to learn $p(r)$, the simulator simulates the honest prover and verifier in the evaluation phase of the polynomial commitment scheme applied to reveal $p(r)$ to the verifier. This completes the description of how the simulator produces a simulated transcript of the verifier's interaction with the prover in the zero-knowledge sum-check protocol. We now explain why the simulated prover messages are distributed identically to those sent by the honest prover in response to the honest verifier.

By the zero-knowledge property of the polynomial commitment scheme, the evaluation proof of the commitment scheme can itself be simulated given $p(r)$ alone, and in particular does not depend on p 's evaluations at any points other than r . This ensures that, conditioned on the values of $g(r)$, ρ , and the prover's messages during the evaluation phase of the polynomial commitment scheme, $q := g + \rho p$ is a random polynomial with the same variable degrees as g , subject to the constraints that $q(r) = g(r) + \rho \cdot p(r)$ and $\sum_{x \in \{0,1\}^\ell} q(x) = G + \rho \cdot P$. Since $f + \rho p$ is a random polynomial subject to the same constraints, the prover messages generated by the simulator are distributed identically to the honest prover's messages in the actual protocol (we omit details of how this assertion is achieved, as it does require modest modifications to the univariate and multilinear polynomial commitment schemes, because p is neither a univariate polynomial nor multilinear).

Costs. When the sum-check protocol is applied in an IP or MIP for circuit-satisfiability, the polynomial g to which it is applied has $\ell \approx 2 \log S$ or $\ell \approx 3 \log S$, where S is either the size of the circuit \mathcal{C} or the number of gates at a single layer of \mathcal{C} (see Sections 4.6 and 7.2). This means that a random polynomial p with the same variable degree as g has at least S^2 coefficients, so even writing p down takes time at least quadratic in S , which is totally impractical. Fortunately, Xie et al. [XZZ⁺19] show that p does not actually need to be a random polynomial of the appropriate variable degrees. Rather, it suffices for p to be a sum of ℓ randomly chosen univariate polynomials s_1, \dots, s_ℓ , one for each of the ℓ variables of g , where the degree of s_i equals $\deg_i(g)$. This ensures that p can be committed to in time $\tilde{O}(\ell)$ using (zero-knowledge variants of) any of the polynomial commitment schemes discussed in this manuscript.

Masking $g(r)$. When the sum-check protocol is applied to a polynomial g in an IP or MIP for circuit- or RICS-satisfiability, allowing the verifier to learn even a single evaluation $g(r)$ of g will violate zero-knowledge, because g itself depends on the witness.

For example, recall that in the MIP for circuit satisfiability of Section 7.2, to check the claim that $\mathcal{C}(x, w) = y$, the prover applies the sum-check protocol exactly once, to the polynomial

$$h_{x,y,Z}(Y) := \tilde{\beta}_{3k}(r, Y) \cdot g_{x,y,Z}(Y).$$

(See Equation (7.2)). Here, Z denotes some polynomial mapping $\{0, 1\}^{\log S}$ to \mathbb{F} , and

$$g_{x,y,Z}(a, b, c) := \tilde{\text{io}}(a, b, c) \cdot (\tilde{\text{I}}_{x,y}(a) - Z(a)) + \tilde{\text{add}}(a, b, c) \cdot (Z(a) - Z(b) + Z(c)) + \tilde{\text{mult}}(a, b, c) \cdot (Z(a) - Z(b) \cdot Z(c)).$$

The honest prover in the MIP sets Z to the multilinear extension \tilde{W} of a correct transcript W for the claim that $\mathcal{C}(x, w) = y$ (where W is viewed as a function mapping $\{0, 1\}^{\log S} \rightarrow \mathbb{F}$).

The key point above is that, since the correct transcript W fully determines the multilinear extension \tilde{W} , and W depends on the witness w , even a single evaluation of \tilde{W} leaks information about w to the verifier. Hence, any zero-knowledge argument system cannot reveal $\tilde{W}(r)$ to the verifier for even a single point r .

Here is a technique for addressing this issue. In a sentence, the idea is to replace $\tilde{W}(r)$ with a slightly higher-degree (randomly chosen) extension polynomial Z of W . This ensures that if the verifier learns a couple of evaluations $Z(r_1), Z(r_2)$ of Z , so long as $r_1, r_2 \notin \{0, 1\}^\ell$, these evaluations are simply independent random field elements and in particular are totally independent of the transcript W .

In more detail, recall further that in argument systems derived from the MIP, the prover uses a polynomial commitment scheme to commit to an extension Z of a correct transcript W . The verifier ignores the commitment until the very end of the sum-check protocol applied to $h_{x,y,Z}$, at which time the verifier needs to evaluate $h_{x,y,Z}$ at a random input $r = (r_1, r_2) \in \mathbb{F}^{\log S} \times \mathbb{F}^{\log S}$. Assuming the verifier can efficiently evaluate $\tilde{\text{io}}, \tilde{\text{I}}, \tilde{\text{add}},$ and $\tilde{\text{mult}}$ on its own, $h_{x,y,Z}(r)$ can be easily computed given $Z(r_1)$ and $Z(r_2)$. The verifier obtains these two values using the evaluation phase of the polynomial commitment scheme.

As discussed above, in the MIP of Section 7.2, the prover sets Z to \tilde{W} , but this does not yield a zero-knowledge argument. Instead, let us modify the protocol as follows to achieve perfect zero-knowledge. First, we insist that the verifier choose the coordinates of r from $\mathbb{F} \setminus \{0, 1\}$ rather than from \mathbb{F} (this has a negligible effect on soundness). Second, we prescribe that the honest prover chooses Z to be a random extension polynomial of the correct transcript W where Z has at least two more coefficients than a multilinear polynomial. For example, we can prescribe that the prover set

$$Z(X_1, \dots, X_{\log S}) := \tilde{W}(X_1, \dots, X_{\log S}) + c_1 X_1(1 - X_1) + c_2 X_2(1 - X_2),$$

where the prover chooses c_1 and c_2 at random. Since $X_1(1 - X_1)$ and $X_2(1 - X_2)$ vanish on inputs in $\{0, 1\}^2$, it is clear that Z extends W . Basic linear algebra implies that for any two points $r_1, r_2 \in \mathbb{F}^{\log S} \setminus \{0, 1\}^{\log S}$, $Z(r_1)$ and $Z(r_2)$ are uniform random field elements, independent of each other and of W . Third, as in the zero-knowledge sum-check protocol described earlier in this section, we insist that the polynomial commitment scheme used to commit to Z is zero-knowledge, meaning that the verifier learns nothing from the commitment or the prover's messages in the evaluation phase of the protocol other than the requested evaluations $Z(r_1)$ and $Z(r_2)$. Fourth, rather than directly applying the sum-check protocol to $h_{x,y,Z}$, we apply the zero-knowledge variant of the sum-check protocol described earlier in this section (with g set to $h_{x,y,Z}$).

The modified argument system clearly remains complete, and it is sound, as the (zero-knowledge) sum-check protocol applied to $h_{x,y,Z}$ confirms with high probability that Z is an extension polynomial of a valid transcript W . It is also perfect zero-knowledge. The simulator is essentially the same as that for the zero-knowledge sum-check protocol. The primary modification of the simulator is that, for $r = (r_1, r_2)$, the

simulator’s oracle query to obtain $g(r)$ is replaced with the following procedure. First, the simulator chooses values $Z(r_1)$ and $Z(r_2)$ to be random field elements, and then derives $g(r)$ based on these values, according to the definition of $g = h_{x,y,Z}$ in Equation (7.2). Second, the simulator simulates the evaluation proof from the polynomial commitment scheme for the evaluations $Z(r_1), Z(r_2)$ using the zero-knowledge property of the commitment scheme.

Costs. The argument system above is essentially the same as the nonzero-knowledge argument system derived from the MIP of Section 7.2, since all we did was replace the multilinear extension \tilde{W} with a slightly higher-degree extension Z . Committing to the extension Z as above does require minor modification of the polynomial commitment schemes covered in this survey, as Z is not multilinear (we omit these details for brevity). However, the modifications add very little cost to the commitment protocol, since Z has only two more coefficients than \tilde{W} .

12.4 Discussion and Comparison

This chapter provided two quite general techniques for transforming a non-zero-knowledge protocol \mathcal{Q} into zero-knowledge one \mathcal{Z} . This allows protocol designers to first design an efficient protocol \mathcal{Q} without having to worry about zero-knowledge, and then apply one of the two transformations to “add” zero-knowledge (hopefully, with minimal concrete overhead or additional cognitive load).

Here is a recap of the first transformation. Suppose in the non-zero-knowledge protocol \mathcal{Q} , all messages from prover to verifier consist of elements of some field \mathbb{F} . Section 12.2 would render the protocol zero-knowledge via the “commit-and-prove” approach as follows: for every field element sent by the prover in \mathcal{Q} , the prover in \mathcal{Z} would instead send a *hiding commitment* to that field element, and then at the end of the protocol the prover in \mathcal{Z} would prove in zero-knowledge (via the proof system of Section 12.1) that the committed values would have caused the \mathcal{Q} verifier to accept.

The downsides of this first transformation are two-fold: first, if (as with Pedersen commitments) the commitment scheme is not binding against quantum adversaries, then \mathcal{Z} will not be post-quantum-sound even if \mathcal{Q} is. Second, as discussed already in Section 12.1.2, verification costs are higher in \mathcal{Z} than in \mathcal{Q} , first because commitments to field elements may be larger than the field elements themselves (thereby increasing proof length) and second because the \mathcal{Z} verifier must effectively run the \mathcal{Q} verifier “on the committed values”, without opening the commitments, and this will further increase proof size and verifier time. For example, each field multiplication that the \mathcal{Q} verifier does may turn into an invocation of Protocol 9 from Section 11.3.2, which requires the prover to send at least 9 extra group elements and the verifier to perform at least nine group exponentiations and several group multiplications, rather than one field multiplication.

While this may appear to be a massive overhead in verifier time and proof length, many non-zero-knowledge protocols \mathcal{Q} will make use of a polynomial commitment scheme. If the commitment scheme is hiding, i.e., it reveals no information to the verifier about the committed polynomial, then the messages sent by the prover of \mathcal{Q} within the scheme do not need to be fed through the commit-and-prove transformation (see the paragraph “Reducing the dependence on witness size below linear” in Section 12.2 for further discussion). In these settings, the verification overhead introduced by the commit-and-prove transformation may be a low-order cost relative to that of the polynomial commitment.

Similarly, from the perspective of the prover’s runtime, the transformation from \mathcal{Q} to \mathcal{Z} does not add much overhead so long as \mathcal{Q} is succinct (i.e., the proof length in \mathcal{Q} is much smaller than the size of the statement being proven). This is because, from the prover’s perspective, all of the cryptographic overhead of the transformation (namely sending commitments to field elements rather than the field elements themselves, and proving in zero-knowledge that the committed values would have caused the \mathcal{Q} verifier to accept) is

applied only to the *verification procedure* in \mathcal{Q} . If this verification procedure is much simpler than statement being proven, this computational overhead should be dwarfed by the cost of simply processing the statement itself, which is a lower bound on the prover’s runtime in \mathcal{Q} .

If \mathcal{Q} is based on the sum-check protocol (Section 4.2), then the second transformation of this chapter, based on masking-polynomials (Section 12.3) can be applied. This has the dual benefits of plausibly preserving post-quantum soundness, and typically adding less overhead than the commit-and-prove-based transformation. With masking polynomials, the main extra cost in the resulting zero-knowledge protocol \mathcal{Z} compared to the non-zero-knowledge protocol \mathcal{Q} is that the prover has to commit to one masking polynomial for each invocation of the sum-check protocol in \mathcal{Q} , and the verifier has to obtain an evaluation of the committed masking polynomial. As described in Section 12.3, these masking polynomials can typically be made very small (of size linear in communication cost of the sum-check protocol, which is typically just logarithmic in the size of the statement being proven).

On the other hand, the masking-polynomial-based transformation is conceptually more complicated and ad hoc than the “commit-and-prove” approach, and accordingly is not as general: it applies only to sum-check-based protocols \mathcal{Q} (though related techniques typically can be used to render other polynomial-based protocols zero-knowledge, such as those in Section 9.3).

Chapter 13

Polynomial Commitment Schemes from Discrete Logarithm or Pairings

Polynomial commitments schemes and a trivial solution. A polynomial commitment scheme is meant to simulate the following idealized process. An untrusted prover \mathcal{P} has in its head a polynomial q (for applications to succinct arguments, we are primarily interested in the cases the q is a univariate polynomial, or a multilinear polynomial). \mathcal{P} sends a complete description of q to the verifier \mathcal{V} (say, a list of all of q 's coefficients over an appropriate basis). \mathcal{V} , having learned q , can evaluate q at any point z of its choosing. In particular, once \mathcal{P} sends the polynomial q to \mathcal{V} , \mathcal{P} cannot go and “change” q based on the point z at which \mathcal{V} wishes to evaluate it. Let us call this procedure, in which \mathcal{P} explicitly sends q to \mathcal{V} , the *trivial polynomial commitment scheme*.

There are three potential issues with the trivial polynomial commitment scheme, two of which involve efficiency considerations.

- In our applications to SNARKs (Chapters 6-9), q may be very large—often as large as the entire statement being proved. So having \mathcal{P} send all coefficients of q to \mathcal{V} will require a huge amount of communication. Hence, using the trivial polynomial commitment does not yield *succinct* arguments.
- \mathcal{V} has to spend time linear in the number of coefficients to compute $q(z)$ (i.e., the trivial polynomial commitment would not yield a *work-saving* argument, meaning one whose verifier is faster than the trivial one that is sent a witness and checks its correctness).
- \mathcal{V} learns the entire polynomial q . This may be incompatible with zero-knowledge (in applications to SNARKs, q typically “encodes” a witness, and hence sending q to \mathcal{V} leaks the entire witness).

Using cryptography, one can hope to address all three issues while achieving the same functionality as the trivial polynomial commitment scheme. Specifically, \mathcal{P} can compute a “compressing” commitment c to q and send only c to the verifier. Compressing means that c is much smaller than q , addressing the first issue above regarding succinctness. Because c is smaller than p itself, c does not bind \mathcal{P} to q in a statistical sense. That is, there will *exist* many different polynomials for which c is a valid commitment, and when the verifier asks \mathcal{P} the evaluation $q(z)$, \mathcal{P} will be able to respond with $p(z)$ for any valid “opening” polynomial p of c . However, it is possible to design polynomial commitment schemes that are *computationally binding*, meaning that any efficient prover (e.g., one unable to solve the discrete logarithm problem, or find a collision in a cryptographic hash function) is unable to respond to any evaluation query z with a quantity other than $q(z)$. More precisely, along with a claimed value v for $q(z)$, the prover will send an *evaluation proof* π .

Computational binding guarantees that any efficient prover will be unable to generate a convincing π unless indeed $v = q(z)$.

We will see polynomial commitment schemes in which π can be checked far faster than what would be required just to read an explicit description of q , thereby addressing the first two issues of the trivial scheme (succinctness and verifier time). We will also see schemes where π reveals nothing about q , and even schemes where, if desired, the verifier does not actually learn the requested evaluation $q(z)$ but rather a *hiding commitment* to $q(z)$. In this manner, the polynomial commitment schemes can support zero-knowledge, leaking no information about q (and the witness it encodes) to the verifier.

Revealing $q(z)$ itself vs. a commitment to $q(z)$. The polynomial commitment schemes we describe in Section 13.1 reveal to the verifier (Pedersen) commitments to the value $v = q(z)$, because this is what is required for their use in the zero-knowledge arguments of Section 12.2. Other zero-knowledge arguments (such as those in Section 12.3) call for v to be revealed explicitly to the verifier. Fortunately, it is easy to modify the commitment schemes of Section 13.1 to reveal v to the verifier, for example by having the prover use Protocol 3 to establish in zero-knowledge that it knows how to open the commitment to value v (see the final paragraph of Section 11.3.1 for details). The pairing-based polynomial commitment scheme in Section 13.2 is described in the setting where the evaluation $v = q(z)$ is revealed explicitly to the verifier.

Polynomial commitments from earlier chapters and how they compare to this chapter. We have previously seen that one way to obtain a polynomial commitment scheme is to combine a PCP or IOP for Reed-Solomon testing with Merkle-hashing (Sections 9.4 and 9.6). Whereas the Merkle-hashing approach only exploited “symmetric key” cryptographic primitives (namely collision-resistant hash functions, combined with the random oracle model to remove interaction), the approaches in this chapter are based on “public key” cryptographic primitives. Such primitives require stronger cryptographic assumptions such as hardness of the discrete logarithm problem in elliptic curve groups. Discussion of the pros and cons of IOP-based polynomial commitments vs. the commitments of this chapter can be found in Section 13.4.

Overview of this chapter’s schemes. Known polynomial commitment schemes tend to be somewhat more general: they enable a prover to commit to any *vector* $u \in \mathbb{F}^n$, and then later prove statements about the inner product of u with any vector $y \in \mathbb{F}^n$ requested by the verifier. In the polynomial commitment scheme, u will be the coefficients of the polynomial q to be committed over an appropriate basis (e.g., the standard monomial basis for univariate polynomial, or the Lagrange basis for multilinear polynomials). Evaluating $q(z)$ is then equivalent to computing the inner product of u with the vector y obtained by evaluating each basis polynomial at z .

For example, if q is univariate, say, $q(X) = \sum_{i=0}^{n-1} u_i X^i$, then for any input z to q $q(z) = \langle u, y \rangle$ where $y = (1, z, z^2, \dots, z^{n-1})$ consists of powers of z , and $\langle u, y \rangle = \sum_{i=0}^{n-1} u_i y_i$ denotes the inner product of u and y . Similarly, if q is multilinear, say $q(X) = \sum_{i=0}^{2^\ell} u_i \chi_i(X)$, where $\chi_1, \dots, \chi_{2^\ell}$ denotes the natural enumeration of the Lagrange basis polynomials¹²⁸, then for $z \in \mathbb{F}_p^\ell$, $q(r) = \langle u, y \rangle$ where $y = \langle \chi_1(z), \dots, \chi_{2^\ell}(z) \rangle$ is the vector of all Lagrange basis polynomials evaluated at z .

Hence, to commit to q , it suffices to commit to the *vector u of coefficients of q* . Then to later reveal (a commitment to) $q(z)$, it suffices to reveal (a commitment to) the inner product of u with the vector y .

¹²⁸See Lemma 3.7 for a definition of the Lagrange basis polynomials. In the natural enumeration, if i has binary representation $i_1, \dots, i_\ell \in \{0, 1\}^\ell$, then $\chi_i(X_1, \dots, X_\ell) = \prod_{j=1}^\ell (X_j^{i_j} + (1 - X_j)(1 - i_j)) = \left(\prod_{j: i_j=1} X_j \right) \left(\prod_{j: i_j=0} (1 - X_j) \right)$.

$$q(z) = 3 + 5z + 7z^2 + 9z^3 + z^4 + 2z^5 + 3z^6 + 4z^7 + 2z^8 + 4z^9 + 6z^{10} + 8z^{11} + 3z^{13} + 6z^{14} + 9z^{15}$$

$$= \begin{array}{|c|c|c|c|} \hline 1 & z^4 & z^8 & z^{12} \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|} \hline 3 & 5 & 7 & 9 \\ \hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 & 6 & 8 \\ \hline 0 & 3 & 6 & 9 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline z \\ \hline z^2 \\ \hline z^3 \\ \hline \end{array}$$

$$= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 3 & 5 & 7 & 9 & 1 & 2 & 3 & 4 & 2 & 4 & 6 & 8 & 0 & 3 & 6 & 9 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & z & z^2 & z^3 & z^4 & z^5 & z^6 & z^7 & z^8 & z^9 & z^{10} & z^{11} & z^{12} & z^{13} & z^{14} & z^{15} \\ \hline \end{array}$$

Figure 13.1: Example of a degree-15 univariate polynomial q expressed via its coefficients over the standard monomial basis. The second line shows that the evaluation $q(z)$ for any input z can be expressed as a vector-matrix-vector product, where the matrix is specified by the coefficients of q , and the two vectors by the evaluation point r . The third line shows $q(z)$ can be equivalently be expressed as an inner product between the coefficient vector of q and an “evaluation vector” consisting of powers of z .

Tensor structure in the evaluation vector. Exactly as in Section 9.6.1, in both the univariate and multilinear cases above, the vector y has a tensor-product structure. Some, but not all, of the polynomial commitment schemes covered in this chapter will exploit this tensor structure (specifically, the schemes in Section 13.1.3 and 13.1.5); the others support inner products of a committed vector with an *arbitrary* vector y .

What we mean by tensor structure is the following. In the univariate case, let $n - 1$ equal the degree of q and let us assume $n = m^2$ is a perfect square, and define $a, b \in \mathbb{F}^m$ as $a := (1, z, z^2, \dots, z^{m-1})$ and $b := (1, z^m, z^{2m}, \dots, z^{m(m-1)})$. If we view y as an $m \times m$ matrix with entries indexed as $(y_{1,1}, \dots, y_{m,m})$, then y is simply the outer product $b \cdot a^T$ of a and b . That is, $y_{i,j} = z^{i \cdot m + j} = b_i \cdot a_j$. Similarly, if q is an ℓ -variate multilinear polynomial, suppose that $2^\ell = m^2$, and let $z_1, z_2 \in \mathbb{F}^{\ell/2}$ denote the first half and second half of $z \in \mathbb{F}^\ell$. Then let χ'_1, \dots, χ'_m denote the natural enumeration of the $(\ell/2)$ -variate Lagrange basis polynomials, and define $a, b \in \mathbb{F}^m$ as $a := (\chi'_1(z_1), \dots, \chi'_m(z_1))$ and $b := (\chi'_1(z_2), \dots, \chi'_m(z_2))$. Then $y = b \cdot a^T$. That is, $y_{i,j} = \chi_{i \cdot m + j}(z) = \chi'_i(z_1) \cdot \chi'_j(z_2) = b_i \cdot a_j$.

In summary, for both univariate and multilinear polynomials q , once the coefficient vector u of q is committed, computing $q(z)$ is equivalent to evaluating the inner product of u with a vector y satisfying $y_{i,j} = b_i \cdot a_j$ for some m -dimensional vectors a, b , where m is the square root of the number of coefficients of q . Equivalently, we can express the inner product of u and y as a vector-matrix-vector product:

$$\langle u, y \rangle = \sum_{i,j=1,\dots,m} u_{i,j} b_i a_j = b^T \cdot u \cdot a, \tag{13.1}$$

where on the right hand side we are viewing u as an $m \times m$ matrix. See Figures 13.1 and 13.2 for examples in both the univariate and multilinear cases.

$$\begin{aligned}
& q(r_1, r_2, r_3, r_4) = \\
& 3(1-r_1)(1-r_2)(1-r_3)(1-r_4) + 5(1-r_1)(1-r_2)(1-r_3)r_4 + 7(1-r_1)(1-r_2)r_3(1-r_4) + 9(1-r_1)(1-r_2)r_3r_4 \\
& + (1-r_1)r_2(1-r_3)(1-r_4) + 2(1-r_1)r_2(1-r_3)r_4 + 3(1-r_1)r_2r_3(1-r_4) + 4(1-r_1)r_2r_3r_4 \\
& + 2r_1(1-r_2)(1-r_3)(1-r_4) + 4r_1(1-r_2)(1-r_3)r_4 + 6r_1(1-r_2)r_3(1-r_4) + 8r_1(1-r_2)r_3r_4 \\
& + 3r_1r_2(1-r_3)r_4 + 6r_1r_2r_3(1-r_4) + 9r_1r_2r_3r_4
\end{aligned}$$

$$= \begin{bmatrix} (1-r_1)(1-r_2) & (1-r_1)r_2 & r_1(1-r_2) & r_1r_2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 & 7 & 9 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 0 & 3 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} (1-r_3)(1-r_4) \\ (1-r_3)r_4 \\ r_3(1-r_4) \\ r_3r_4 \end{bmatrix}$$

Figure 13.2: Example of a 4-variate multilinear polynomial q expressed via its coefficients over the Lagrange basis (see Lemma 3.7). The evaluation $q(r)$ for any input $r = (r_1, r_2, r_3, r_4) \in \mathbb{F}^4$ can be expressed as a vector-matrix-vector product, where the matrix is specified by the coefficients of q , and the two vectors by the evaluation point r .

13.1 Polynomial Commitments from Hardness of Discrete Logarithm

13.1.1 A Zero-Knowledge Scheme with Linear Size Commitments

We begin by describing a scheme that does not improve over the costs of the trivial polynomial commitment scheme, but does render it zero-knowledge. That is, is the prover’s commitment to q is as large of q itself, and given the commitment to q , the verifier on its own can derive a commitment to $q(z)$ for any input z of the verifier’s choosing.

Recall that in a Pedersen commitment (Section 11.3) over group \mathbb{G} of prime order p with generators g, h , a commitment to a value $m \in \mathbb{F}_p$ is $c \leftarrow h^m \cdot g^r$ for a value $r \in \{0, \dots, p-1\}$ randomly chosen by the committer. Pedersen commitments are perfectly hiding and computationally binding.

Commitment Phase. To commit to q , rather than the prover sending each entry of the coefficient vector u to the verifier “in the clear” as in the trivial scheme, \mathcal{P} sends a Pedersen commitment to each entry of u . Pedersen commitments are hiding, so this reveals to the receiver nothing at all about u .

Evaluation Phase. Let y be the vector such that $q(z) = \langle u, y \rangle = \sum_i u_i y_i$. Since the verifier knows y and has a commitment to each entry u_i of u , using the homomorphism property of Pedersen commitments, the verifier can on its own derive a commitment c to $\sum_i u_i y_i$. \mathcal{P} can prove in zero-knowledge that it knows how to open the commitment c via Protocol 7 (Section 11.3.1).

13.1.2 Constant Size Commitments But Linear-Size Evaluation Proofs

In the commitment scheme of Section 13.1.1, the commitment was as big as the polynomial being committed. In this section, we give a scheme that reduces the commitment size to constant (one group element). However, evaluation proofs (and hence also verification time) will become very large—as big as the polynomial being committed.¹²⁹

¹²⁹The number of public parameters for the scheme of this section is also very large, consisting of n randomly chosen group elements g_1, \dots, g_n , where n is the length of the coefficient vector u . But in the random oracle model g_i can be chosen via a random oracle by evaluating the random oracle at input i , in which case the public parameter size is constant.

Commitment Phase. Assume that n generators g_1, \dots, g_n for \mathbb{G} are chosen at random from \mathbb{G} . To commit to $u \in \mathbb{F}_p^n$, committer will pick a random value $r_u \in \{0, 1, \dots, |\mathbb{G}| - 1\}$ and send the value $\text{Com}(u; r_u) := h^{r_u} \cdot \prod_{i=1}^n g_i^{u_i}$. This quantity is often referred to as a *generalized Pedersen commitment*, or a *Pedersen vector commitment* (a standard Pedersen commitment is equivalent to a generalized Pedersen commitment when $n = 1$). Note that Pedersen vector commitments are homomorphic: given two commitments c_u, c_w to two vectors u and w in \mathbb{F}_p^n , and any two scalars $a_1, a_2 \in \mathbb{F}_p$, one can compute a commitment to the linear combination $a_1 u + a_2 w$, as $c_u^{a_1} \cdot c_w^{a_2}$.

Pedersen *vector* commitments should be contrasted with the scheme of Section 13.1.1, which committed to the vector $u \in \mathbb{F}_p^n$ by sending a different Pedersen commitment for each entry of u (using the same public group generator g for all n commitments). This was not a compressing commitment. Pedersen vector commitments compute a different Pedersen commitment $g_i^{u_i}$ for each entry u_i of u (but without a blinding factor, see Footnote 123), with each commitment using a different group generator $g_i \in \mathbb{G}$. But rather than sending all n commitments to the verifier, they are all “compressed” into a single commitment using the group operation of \mathbb{G} (and then the result is blinded by the factor h^{r_u}).

Evaluation Phase. Recall that evaluating a committed polynomial q at input z is equivalent to computing $\langle u, y \rangle$ for the coefficient vector u and a vector y derived from z . Suppose we are given a commitment $c_u = \text{Com}(u, r_u)$, a public query vector y , and a commitment $c_v = \text{Com}(v, r_v) = g_1^{v_1} \cdot h^{r_v}$ where $v = \langle u, y \rangle$, and the committer knows r_u and r_v but the verifier does not. The committer wishes to prove in zero-knowledge that it knows an opening u of c_u and v of c_v such that $\langle u, y \rangle = v$, as unless the prover can break the binding property of the commitments, this is equivalent to establishing that $q(z) = v$.

As with Protocol 7 (see the end of Section 11.3.1), directly opening the commitments to u and v would enable the verifier to easily check that $v = \langle u, y \rangle$, but would violate zero-knowledge. So instead the prover opens *derived* commitments, with both the prover and verifier contributing randomness to the derived commitments in a manner such that the derived commitments satisfy the same property that the prover claims is satisfied by the original commitments.

In more detail, first the committer samples a random n -dimensional vector d with entries in $\{0, \dots, p-1\}$ and two random values $r_1, r_2 \in \{0, \dots, p-1\}$. The committer sends two values $c_1, c_2 \in \mathbb{G}$ claimed to equal $\text{Com}(d, r_1)$ and $\text{Com}(\langle d, y \rangle, r_2)$. The verifier responds with a random challenge $e \in \{0, \dots, p-1\}$. The prover responds with three quantities $u', r_{u'}, r_{v'} \in \{0, \dots, p-1\}$ claimed to respectively equal the following (with all arithmetic done modulo p):

$$e \cdot u + d \in \{0, \dots, p-1\}^n, \tag{13.2}$$

$$e \cdot r_u + r_1 \in \{0, \dots, p-1\}, \tag{13.3}$$

$$e \cdot r_v + r_2 \in \{0, \dots, p-1\}. \tag{13.4}$$

Finally, the verifier checks that $c_u^e \cdot c_1 = \text{Com}(u', r_{u'})$ and $c_v^e \cdot c_2 = \text{Com}(\langle u', y \rangle, r_{v'})$.

This protocol can be proved complete, special-sound, and perfect honest-verifier zero knowledge in a manner very similar to Protocol 7. Before writing out the formal analysis, we explain how each step of the protocol here is in direct analogy with each step of Protocol 7.

In Protocol 7, the prover’s first message contained a commitment to a random value $d \in \{0, \dots, p-1\}$. Here, since we are dealing with vector commitments, the prover’s first message contains a commitment to

Protocol 11 Evaluation phase of the polynomial commitment scheme of Section 13.1.2. If the committed polynomial is q and the evaluation point is z , $u \in \mathbb{F}_p^n$ denotes the coefficient vector of q (which is assumed to be defined over field \mathbb{F}_p for prime p) and $y \in \mathbb{F}_p^n$ is a vector such that $q(z) = \langle u, y \rangle = \sum_{i=1}^n u_i \cdot y_i$.

- 1: Let \mathbb{G} be a (multiplicative) cyclic group of prime order p over which the Discrete Logarithm relation is hard, with randomly chosen generators h, g_1, \dots, g_n and g .
- 2: Let $c_u = \text{Com}(u; r_u) := h^{r_u} \cdot \prod_{i=1}^n g_i^{u_i}$ and $c_v = \text{Com}(v; r_v) = g^v h^{r_v}$. Prover knows u, r_u, v , and r_v . Verifier only knows $c_u, c_v, h, g_1, \dots, g_n$, and g .
- 3: Prover picks $d \in \{0, \dots, p-1\}^n$ and $r_1, r_2 \in \{0, \dots, p-1\}$ and sends to verifier $c_d := \text{Com}(d, r_1)$ and $c_{\langle d, y \rangle} := \text{Com}(\langle d, y \rangle, r_2)$.
- 4: Verifier sends challenge e chosen at random from $\{0, \dots, |\mathbb{G}| - 1\}$.
- 5: Let $u' \leftarrow u \cdot e + d$ and $r_{u'} \leftarrow r_u \cdot e + r_1$, and let $c_{u'} \leftarrow \text{Com}(u', r_{u'})$. While Verifier does not know u' and $r_{u'}$, Verifier can derive $c_{u'}$ unaided from c_u and c_d using additive homomorphism, as $c_u^e \cdot c_d$.
- 6: Similarly, let $v' \leftarrow v \cdot e + \langle d, y \rangle = \langle u', y \rangle$ and $r_{v'} \leftarrow r_v \cdot e + r_2$, and let $c_{v'} \leftarrow \text{Com}(v', r_{v'})$. While Verifier does not know v' and $r_{v'}$, Verifier can derive $c_{v'}$ unaided from c_v and $c_{\langle d, y \rangle}$ using additive homomorphism, as $c_v^e \cdot c_{\langle d, y \rangle}$.
- 7: Prover sends $(u', r_{u'})$ and $r_{v'}$ to Verifier.
- 8: Verifier checks that $(u', r_{u'})$ is valid opening information for $c_{u'}$, and that $(\langle u', y \rangle, r_{v'})$ is valid opening information for $c_{v'}$. Equivalently, Verifier checks that:

$$h^{r_{u'}} \cdot \prod_{i=1}^n g_i^{u'_i} = c_{u'}$$

and

$$h^{r_{v'}} \cdot g^{\langle u', y \rangle} = c_{v'}.$$

a random vector d with entries in $\{0, \dots, p-1\}$. Since this protocol is meant to establish not only that the prover knows how to open the commitment to vector u , but also that the prover knows how to open the second commitment to $\langle u, y \rangle$, the protocol here also has the prover send a second commitment, to $\langle d, y \rangle$.

In Protocol 7, after the verifier sent a random challenge e , the prover responded with an opening of the commitment to $e \cdot m + r_1$ that can be derived homomorphically from the commitments to m and d . Analogously, here the prover responds with opening information for the derived commitments to the vector $e \cdot u + d$ (that opening information is specified via Equations (13.2) and (13.3)) and to the value $\langle eu + d, y \rangle$ (the opening information is $(\langle u', y \rangle, r_{v'})$, where if the prover is honest, u' is specified as per Equation (13.2) and $r_{v'}$ is specified as per Equation (13.4)). In both protocols, the verifier simply checks that the opening information for both commitment(s) is valid. Effectively, the verifier confirms that the derived commitments, to vector $u' = eu + d$ and to value $\langle u', y \rangle$, satisfy the same relationship that the prover claims holds between the original committed vector u and value v , namely that the latter equals the inner product of the former with vector y .

Completeness, special soundness, and zero-knowledge. Completeness is clear by inspection of Protocol 11. For special soundness, let $(c_d, c_{\langle d, y \rangle})$ be the first message sent by the prover, and let $((c_d, c_{\langle d, y \rangle}), e, (u^*, c_{u^*}, r^*))$ and $((c_d, c_{\langle d, y \rangle}), e', (\hat{u}, c_{\hat{u}}, \hat{r}))$ be two accepting transcripts. Owing to the transcripts passing the two tests performed by the verifier in Step 8 of Protocol 11), this means that:

$$h^{r_{u^*}} \cdot \prod_{i=1}^n g_i^{u_i^*} = c_u^e \cdot c_d, \quad (13.5)$$

$$h^{r_{\hat{u}}} \cdot \prod_{i=1}^n g_i^{\hat{u}_i} = c_u^{e'} \cdot c_d, \quad (13.6)$$

$$h^{r^*} \cdot g^{\langle u^*, y \rangle} = c_v^e \cdot c_{\langle d, y \rangle}. \quad (13.7)$$

$$h^{\hat{r}} \cdot g^{\langle \hat{u}, y \rangle} = c_v^{e'} \cdot c_{\langle d, y \rangle}. \quad (13.8)$$

Let

$$\bar{u} := (u^* - \hat{u}) \cdot (e - e')^{-1} \pmod{|\mathbb{G}|},$$

$$r_{\bar{u}} := (r_{u^*} - r_{\hat{u}}) \cdot (e - e')^{-1} \pmod{|\mathbb{G}|},$$

and

$$r_{\bar{v}} := (r^* - \hat{r}) \cdot (e - e')^{-1} \pmod{|\mathbb{G}|}$$

Dividing Equation (13.5) by Equation (13.6) implies that

$$h^{\bar{v}} \cdot \prod_{i=1}^n g_i^{\bar{u}_i} = c_u,$$

and dividing Equation (13.7) by Equation (13.8) implies that:

$$h^{\bar{r}} \cdot g^{\langle \bar{u}, y \rangle} = c_v.$$

That is, $(\bar{u}, r_{\bar{u}})$ is an opening of c_u and $(\langle \bar{u}, y \rangle, r_{\bar{v}})$ is an opening of c_v , and the two openings satisfy the claimed relationship that the value committed by c_v is the inner product of the vector committed by c_u with y .

To establish honest-verifier perfect zero-knowledge, consider the following simulator. To generate an accepting transcript $((c_d, c_{\langle d, y \rangle}), e, (u', r_{u'}, r_{v'}))$, the simulator proceeds as follows. First, it selects the verifier's challenge e at random from $\{0, \dots, p-1\}$, and then picks a vector u' at random from $\{0, 1, \dots, p-1\}^n$ and $r_{u'}, r_{v'}$ at random from $\{0, 1, \dots, p-1\}$. Finally, it chooses c_d and $c_{\langle d, y \rangle}$ to be the unique values that yield an accepting transcript, i.e., c_d is set to $c_u^{-e} \cdot h^{r_{u'}} \cdot \prod_{i=1}^n g_i^{u'_i}$ and $c_{\langle d, y \rangle}$ is set to $c_v^{-e} \cdot h^{r_{v'}} \cdot g^{\langle u', y \rangle}$. These choices of c_d and c_v are specifically chosen to ensure that the generated transcript is an accepting one. One can show that the distribution over accepting transcripts generated by this simulator is equal to the distribution generated by the honest verifier interacting with the honest prover by establishing a one-to-one correspondence between transcripts that the simulator outputs with transcripts generated by the honest verifier interacting with the honest prover (details omitted for brevity).

Costs. The commitment consists of a single group element. The computational cost of computing the commitment is performing n group exponentiations. Naively performing each exponentiation independently using repeated squaring requires $O(\log |\mathbb{G}|)$ group multiplications per exponentiation, which implies $\Theta(n \log |\mathbb{G}|)$ group multiplications in total. However, Pippenger's multi-exponentiation algorithm [Pip80] can reduce this quantity by a factor of $(\log(n) + \log \log |\mathbb{G}|)$.¹³⁰

In the evaluation phase, the proof consists of $n+2$ numbers in $\{0, \dots, p-1\}$ that can be computed with $O(n)$ field operations in \mathbb{F}_p in total. The verification procedure requires the verifier to perform $O(n)$ group exponentiations, which can be performed using Pippenger's algorithm in the time bound described in the previous paragraph.

13.1.3 Trading Off Commitment Size and Proof Size

Recall that the polynomial commitment scheme of the previous section had very small commitments (1 group element), but large proofs-of-evaluation ($\Theta(n)$ group elements).

In this section, we show how to exploit tensor-structure in the vector y (captured in Equation (13.1)) to reduce the size of the proof in the evaluation phase of the polynomial commitment scheme of the previous section, at the cost of increasing the commitment size. For example, we can set both the commitment size and the evaluation proof size to $\Theta(\sqrt{n})$ group elements. This technique was presented in the context of multilinear polynomials in a system called Hyrax [WTS⁺18], building directly on a univariate polynomial commitment scheme given in [BCC⁺16].

Commitment Phase. Recall that u denotes the coefficient vector of the polynomial q to which the committer wishes to commit, and as per Equation (13.1), we view u as an $m \times m$ matrix. Letting $u_j \in \mathbb{F}^m$ denote the j th column of u , the committer picks random numbers $r_1, \dots, r_m \in \{0, \dots, p-1\}$ and sends a set of vector commitments $c_1 = \text{Com}(u_1, r_1), \dots, c_m = \text{Com}(u_m, r_m)$, one for each column. Here,

$$\text{Com}(u_j, r_j) = h^{r_j} \cdot \prod_{k=1}^m g_k^{u_{j,k}}$$

for public parameters $g_1, \dots, g_m \in \mathbb{G}$. Hence, compared to the previous section, we have increased the size of the commitment for u from 1 group element to m group elements. Rather than applying the vector-

¹³⁰A *multi-exponentiation* in a multiplicative group is a product of powers of elements of the group.

commitment scheme of the previous section to one vector of length m^2 , we applied it m times, to vectors of length m .

Evaluation Phase. When the verifier asks the committer to provide a commitment to $q(z)$ for a verifier-selected input z , the prover sends a commitment c^* to $q(z) = \langle u, y \rangle = b^T \cdot u \cdot a$, where the m -dimensional vectors b and a are as in Equation (13.1) and are known to both the prover and verifier. Using the additive homomorphism of the commitment scheme, the verifier can on its own compute a commitment to the vector $u \cdot a$, namely $\prod_{j=1}^m \text{Com}(u_j)^{a_j}$. At this point, the prover needs to prove that c^* is a commitment to $b^T \cdot (u \cdot a) = \langle b, u \cdot a \rangle$. Since the verifier has derived a commitment to the vector $u \cdot a$, this is exactly an instance of the problem that the protocol of the previous section was designed to solve, using a proof of size m .

In summary, we have given a (public-coin) commitment scheme for univariate and multilinear polynomials in which the commitment size proof length in the evaluation phase, and total verifier time are equal to the square root of the number of coefficients of the polynomial.

13.1.4 Bulletproofs

In this section, we give a scheme in which the commitment size is constant and the proof length in the evaluation phase is logarithmic in the number of coefficients of the polynomial. However, the verifier's runtime to process the proof is *linear* in the number of coefficients. Compared to the commitment scheme of Section 13.1.2, this is a strict improvement because the proof length in the evaluation phase is logarithmic as opposed to linear in the length of the coefficient vector. Compared to the scheme of Section 13.1.3, the verifier's runtime is worse (linear rather than proportional to the square root of the number of coefficients), but the communication cost is much better (logarithmic as opposed square root).

The scheme of this section is a variant of a system called Bulletproofs [BBB⁺18], which itself directly builds on a univariate polynomial commitment scheme given in [BCC⁺16]. Our presentation draws substantially on a perspective developed in [BDFG20].

13.1.4.1 Warm-up: Proof of Knowledge for Opening of Vector Commitment

Before presenting the polynomial commitment in full, we start with a warmup that illustrates the key ideas of the full Bulletproofs polynomial commitment scheme. Specifically, the warmup is a protocol enabling the prover to establish that it knows how to open a generalized Pedersen commitment to a vector $u \in \mathbb{F}_p^n$.

Notational changes for this section. Recall that a generalized Pedersen commitment to u is $\text{Com}(u; r_u) := h^{r_u} \cdot \prod_{i=1}^n g_i^{u_i}$ where r_u is chosen at random by the committer, and the g_i 's are public generators in \mathbb{G} . To further simplify the presentation, let us omit the blinding factor h^{r_u} from the vector commitment, so that we now define $\text{Com}(u) := \prod_{i=1}^n g_i^{u_i}$ (the resulting commitment scheme without the blinding factor is still computationally binding, but it is only computationally rather than perfectly hiding, see Footnote 123 in Section 11.3).

For the remainder of this section, we write \mathbb{G} as an additive rather than multiplicative group. This is because, by doing so, we can think of $\text{Com}(u) = \sum_{i=1}^n u_i \cdot g_i$ as the *inner product* between u and the vector $\mathbf{g} = (g_1, \dots, g_n)$ of public group generators, and hence we denote $\sum_{i=1}^n u_i \cdot g_i$ as $\langle u, \mathbf{g} \rangle$.¹³¹

¹³¹Strictly speaking, referring to $\sum_{i=1}^n u_i g_i$ as an inner product is a misnomer because the u_i 's are integers in $\{0, 1, \dots, p-1\}$ while the g_i 's are elements of the group \mathbb{G} of size p , but we ignore this and write $\text{Com}(u) = \langle u, \mathbf{g} \rangle$ for the remainder of this section.

Overview of the protocol. The protocol is vaguely reminiscent of the IOP-based polynomial commitment scheme FRI (Section 9.4), in the following sense. At the start of the protocol, the prover has sent a commitment c_u to a vector u of length n . The protocol proceeds in $\log_2 n$ rounds, where in each round i the verifier sends the prover a random field $\alpha_i \in \mathbb{F}_p$, and α_i is used to “halve the length of the committed vector”. In more detail, at the start of each round $i = 1, 2, \dots, \log_2 n$, the prover has sent a commitment $c_{u^{(i)}}$ to some vector $u^{(i)}$ of length $n \cdot 2^{-(i-1)}$, and the prover must establish that it knows a vector $u^{(i)}$ such that $\langle u^{(i)}, \mathbf{g}^{(i)} \rangle = c_{u^{(i)}}$ (when $i = 1$, $u^{(i)} = u$ and $\mathbf{g}^{(i)} = \mathbf{g}$).

For notational brevity let us fix a round i and accordingly drop the superscript (i) , simply writing u , \mathbf{g} , and c_u . The idea of the protocol is to break u and \mathbf{g} into two halves, writing $u = u_L \circ u_R$ and $\mathbf{g} = \mathbf{g}_L \circ \mathbf{g}_R$, where \circ denotes concatenation. Then

$$\langle u, \mathbf{g} \rangle = \langle u_L, \mathbf{g}_L \rangle + \langle u_R, \mathbf{g}_R \rangle. \quad (13.9)$$

A Key Equality. The following simple equality is an immediate application of the distributive law, and holds for any $\alpha \in \mathbb{F}_p$:

$$\begin{aligned} \langle \alpha u_L + \alpha^{-1} u_R, \alpha^{-1} \mathbf{g}_L + \alpha \mathbf{g}_R \rangle &= \langle \alpha u_L, \alpha^{-1} \mathbf{g}_L \rangle + \langle \alpha^{-1} u_R, \alpha \mathbf{g}_R \rangle + \langle \alpha u_L, \alpha \mathbf{g}_R \rangle + \langle \alpha^{-1} u_R, \alpha^{-1} \mathbf{g}_L \rangle \\ &= (\langle u_L, \mathbf{g}_L \rangle + \langle u_R, \mathbf{g}_R \rangle) + \alpha^2 \langle u_L, \mathbf{g}_R \rangle + \alpha^{-2} \langle u_R, \mathbf{g}_L \rangle \\ &= \langle u, \mathbf{g} \rangle + \alpha^2 \langle u_L, \mathbf{g}_R \rangle + \alpha^{-2} \langle u_R, \mathbf{g}_L \rangle, \end{aligned} \quad (13.10)$$

where the final equality uses Equation (13.9).

A Protocol Exploiting the Key Equality. Recall that at the start of the round, the prover has already sent a value c_u claimed to equal $\langle u, \mathbf{g} \rangle$. If u and \mathbf{g} both have length 1, then establishing that the prover knows a u such that $\langle u, \mathbf{g} \rangle = c_u$ is equivalent to establishing knowledge of a discrete logarithm of c_u to base \mathbf{g} , which the prover can achieve by simply sending u to \mathcal{V} .¹³² Otherwise, the protocol proceeds as follows: The prover starts by sending values v_L, v_R claimed to equal the “cross terms” $\langle u_L, \mathbf{g}_R \rangle$ and $\langle u_R, \mathbf{g}_L \rangle$. At that point the verifier chooses $\alpha \in \mathbb{F}_p$ at random and sends it to the prover.

Let $u' = \alpha u_L + \alpha^{-1} u_R$ and $\mathbf{g}' = \alpha^{-1} \mathbf{g}_L + \alpha \mathbf{g}_R$, and let $c_{u'} = c_u + \alpha^2 v_L + \alpha^{-2} v_R$. These values are specifically defined so that if v_L and v_R are as claimed, then $\langle u', \mathbf{g}' \rangle = c_{u'}$. Furthermore, the verifier can compute \mathbf{g}' and $c_{u'}$ given c_u , α , v_L , and v_R . Accordingly, the next round of the protocol is then meant to establish that the prover indeed knows a vector u' such that

$$\langle u', \mathbf{g}' \rangle = c_{u'}. \quad (13.11)$$

This is exactly the type of claim that the protocol was meant to establish, but on vectors of length $n/2$ rather than n , so the protocol verifies this claim recursively. See Protocol 12 for pseudocode.

Costs. It is easy to see that u' and \mathbf{g}' have half the length of u and \mathbf{g} , and hence the protocol terminates after $\log_2 n$ rounds, with only two group elements sent by the prover to the verifier in each round. Both the prover and verifier’s runtimes are dominated by the time required to update the generator vector in each round. Specifically, to compute \mathbf{g}' from \mathbf{g} in each round, the verifier performs a number of group

¹³²For simplicity, we do not concern ourselves during this warmup with designing a zero-knowledge protocol; if we did want to achieve zero-knowledge, we would use Schnorr’s protocol (Section 11.2.2) for the prover to establish knowledge of the discrete logarithm of c_u . See Section 13.1.4.2 for details.

Protocol 12 A public-coin zero-knowledge argument of knowledge of an opening for a generalized Pedersen commitment c_u to a vector u of length n . The protocol consists of $\log_2 n$ rounds and 2 group elements communicated from prover to verifier per round, and satisfies knowledge soundness assuming hardness of the discrete logarithm problem. For simplicity, we omit the blinding factor from the Pedersen commitment to u and treat the group \mathbb{G} over which the commitments are defined as an additive group.

- 1: Let \mathbb{G} be an *additive* cyclic group of prime order p over which the Discrete Logarithm relation is hard, with vector of generators $\mathbf{g} = (g_1, \dots, g_n)$.
- 2: Input is $c_u = \text{Com}(u) := \sum_{i=1}^n u_i g_i$. Prover knows u , Verifier only knows c_u, g_1, \dots, g_n .
- 3: If $n = 1$, Prover sends u to the verifier and the verifier checks that $u g_1 = c_u$.
- 4: Otherwise, write $u = u_L \circ u_R$ and $\mathbf{g} = \mathbf{g}_L \circ \mathbf{g}_R$. Prover sends v_L, v_R claimed to equal $\langle u_L, \mathbf{g}_R \rangle$ and $\langle u_R, \mathbf{g}_L \rangle$.
- 5: Verifier responds with a randomly chosen $\alpha \in \mathbb{F}_p$.
- 6: Recurse on commitment $c_{u'} := c_u + \alpha^2 v_L + \alpha^{-2} v_R$ to vector $u' = \alpha u_L + \alpha^{-1} u_R$ of length $n/2$, using the vector of group generators $\mathbf{g}' := \alpha^{-1} \mathbf{g}_L + \alpha \mathbf{g}_R$.

exponentiations proportional to the length of \mathbf{g} , which means the total number of group exponentiations is $O(n + n/2 + n/4 + \dots + 1) = O(n)$.¹³³ Hence's the verifier's runtime over the entire protocol is proportional to the time required to perform $O(n)$ group exponentiations.

Completeness and intuition for knowledge soundness. The protocol is clearly complete, i.e., if the prover is honest (meaning that c_u indeed equals $\langle u, \mathbf{g} \rangle$ and v_L and v_R are as claimed), then indeed Equation (13.11) holds.

To explain the intuition for why knowledge soundness holds, let us assume for a moment that the prover *does* know a vector u such that $c_u = \langle u, \mathbf{g} \rangle$, but the prover sends values v_L and v_R that are *not* equal to $\langle u_L, \mathbf{g}_R \rangle$ and $\langle u_R, \mathbf{g}_L \rangle$. Then, as we explain in the following paragraph, with high probability over the choice of α , Equation (13.11) will fail to hold. In this event, it is not clear how the prover will find a vector whose inner product with \mathbf{g}' equals $c_{u'}$.

That Equation (13.11) fails to hold with high probability over the choice of α follows by the following reasoning. Let Q be the degree-4 polynomial

$$Q(\alpha) = \alpha^2 c_{u'} = \alpha^2 c_u + \alpha^4 v_L + v_R$$

and

$$P(\alpha) = \alpha^2 \cdot \langle u', \mathbf{g}' \rangle = \alpha^2 c_u + \alpha^4 \langle u_L, \mathbf{g}_R \rangle + \langle u_R, \mathbf{g}_L \rangle.$$

If v_L and v_R are not equal to $\langle u_L, \mathbf{g}_R \rangle$ and $\langle u_R, \mathbf{g}_L \rangle$, then Q and P are not the same polynomial. Since they both have degree at most 4, with probability at least $4/p$ over the random choice of α , $Q(\alpha) \neq P(\alpha)$. In this event, $c_{u'} \neq \langle u', \mathbf{g}' \rangle$, and hence the prover is left to prove a false claim in the next round.

The above line of reasoning suggests that a prover who knows an opening u of c_u should not be able to convince the verifier to accept with non-negligible probability if the prover does not behave as prescribed in each round (i.e., if sending commitments to values v_L and v_R not equal to $\langle u_L, \mathbf{g}_R \rangle$ and $\langle u_R, \mathbf{g}_L \rangle$). But to formally establish knowledge soundness, we must show that given any prover \mathcal{P} that convinces the verifier to accept with non-negligible probability, there is an efficient algorithm to extract an opening u of c_u from \mathcal{P} . This requires a more involved analysis.

¹³³The terminology “group exponentiation” here, while standard, may be confusing because in this section we are referring to \mathbb{G} as an additive group, while the terminology refers to a multiplicative group. In the additive group notation of this section, we are referring to taking a group element and multiplying it by α or α^{-1} . The same operation in multiplicative group notation would be denoted by raising the group element to the power α or α^{-1} , hence our use of the term group exponentiation.

Formal proof of knowledge soundness. The analysis establishing knowledge soundness follows the following standard two-step paradigm.

Step 1: Another forking lemma. First, we argue that there is a polynomial-time extraction algorithm \mathcal{E} that, given any prover \mathcal{P} for Protocol 12 that convinces the verifier to accept with non-negligible probability, constructs a 3-transcript-tree \mathcal{T} for the protocol with non-negligible probability. Here, a 3-transcript tree is a collection of $|\mathcal{T}| = 3^{\log_2 n} \leq n^{1.585}$ accepting transcripts for the protocol, with the following relationship between the prover messages and verifier challenges in each transcript.

The transcripts correspond to the leaves of a complete tree where each non-leaf node has 3 children. Each edge of the tree is labeled by a verifier challenge, and each non-leaf node is associated with a prover message (v_L, v_R) . That is, if an edge of the tree connects a node at distance i from the root to a node at distance $i + 1$, then the edge is labelled by a value for the i th message that the verifier sends to the prover in Protocol 12. It is required that (a) no two edges of the tree are assigned the same label and (b) for each transcript at a leaf of the tree, the verifier’s challenges in that transcript are given by the labels assigned to the edges of the root-to-leaf path for that leaf, and the prover’s messages in the transcript are given by the prover responses associated with the nodes along the path.

The fact that \mathcal{E} can extract a 3-transcript tree from \mathcal{P} follows from a generalization of the reasoning used to analyze the knowledge soundness of commit-and-prove zero-knowledge arguments for circuit satisfiability (see Footnote 126 in Section 12.1). It is referred to as a *forking lemma* because it is proved via an argument closely related to the (different) forking lemma from Section 11.2.3, which was used to analyze the knowledge soundness of the non-interactive protocol obtained by applying the Fiat-Shamir transformation to any Σ -protocol. We provide a formal statement and proof of this result below; our proof follows [BCC⁺16, Lemma 1]. The idea is to generate the first leaf of the tree by running the prover and verifier once to (hopefully) generate an accepting transcript. Then generate that leaf’s sibling by rewinding the prover until just before the verifier sends its last challenge, and restart the protocol with a fresh random value for the verifier’s final challenge. Then to generate the next leaf, rewind the prover again until just before the verifier sends its second to last challenge, and restart the protocol from that point with a fresh random value for the verifier’s second to last challenge. And so on. Some complications arise to account for the possibility that the prover sometimes fails to convince the verifier to accept, and the (unlikely) possibility that this process leads to two edges labeled with the same value.

Theorem 13.1. *There is a probabilistic extractor algorithm \mathcal{E} satisfying the following property. Given the ability to repeatedly run and rewind a prover \mathcal{P} for Protocol 12 that causes the verifier to accept with probability at least ϵ for some non-negligible quantity ϵ , \mathcal{E} runs in expected time at most $\text{poly}(n)$, and \mathcal{E} outputs a 3-transcript tree \mathcal{T} for Protocol 12 with probability at least $\epsilon/2$.*

Proof. \mathcal{E} is a recursive procedure that constructs \mathcal{T} in depth-first fashion. Specifically, \mathcal{E} takes as input the identity of a node j in \mathcal{T} , as well as the verifier challenges associated with the edges along the path in \mathcal{T} connecting j to the root, and the prover messages associated with the nodes along that path. \mathcal{E} then (attempts to) produce the subtree of \mathcal{T} rooted at j . (In the very first call to \mathcal{E} , j is the root node of \mathcal{T} , so in this case there are no edges or nodes along the path of the j to the root, i.e., itself).

If j is a leaf node, the input to \mathcal{E} specifies a complete transcript for Protocol 12, so \mathcal{E} simply outputs the transcript if it is an accepting transcript, and otherwise it outputs “fail”.

If j is not a leaf node of \mathcal{T} , then the input to \mathcal{E} specifies a partial transcript for Protocol 12 (if j has distance ℓ from the root, then the partial transcript specifies the prover messages and verifier challenges from the first ℓ rounds of Protocol 12). The first thing \mathcal{E} does is associate a prover message with j by

“running” \mathcal{P} on the partial transcript to see how \mathcal{P} would respond to the most recent verifier challenge in this partial transcript.

Second, \mathcal{E} attempts to construct the subtree rooted at the left-most subchild of j , which we denote by j' . Specifically, \mathcal{E} chooses a random verifier challenge to assign the edge (j, j') of \mathcal{T} , and then calls itself recursively on j' . If \mathcal{E} 's recursive call on j' returns “fail” (i.e., it fails to generate the subtree of \mathcal{T} rooted at j'), then \mathcal{E} halts and outputs “fail”. Otherwise, \mathcal{E} proceeds to generate the subtrees of the remaining two children j'' and j''' of j by assigning fresh random verifier challenges to the edges connecting j to those nodes and calling itself recursively on j'' and j''' until it successfully generates these two subtrees (this may require many repetitions of the recursive calls, as \mathcal{E} will simply keep calling itself on j'' and j''' until it finally succeeds in generating these two subtrees).

Expected running time of \mathcal{E} . Recall that when \mathcal{E} is called on a non-leaf node j , it recursively calls itself once on the first child j' of j in an attempt to construct the subtree rooted at j' , and then continues to construct the subtrees rooted at its other two children only if the recursive call on j' succeeds. Let ϵ' denote this probability. Then the expected number of recursive calls is $1 + \epsilon' \cdot 2/\epsilon' = 3$. Here, the first term, 1, comes from the first recursive call, on j' . The first factor of ϵ' in the second term denotes the probability that \mathcal{E} does not halt after the first recursive call. Finally, the factor $2/\epsilon'$ captures the expected number of times \mathcal{E} must be called on j'' and j''' before it succeeds in constructing the subtree rooted at these nodes (as $1/\epsilon'$ is the expected value of a geometric random variable with success probability ϵ'). Meanwhile, when \mathcal{E} is called on a leaf node, it simply checks whether or not the associated transcript is an accepting transcript, which requires $\text{poly}(n)$ time. We conclude that the total runtime of \mathcal{E} is proportional to the number of leaves (which is $3^{\log_2 n} \leq O(n^{1.585})$), times the runtime of the verifier in Protocol 12, which is clearly $\text{poly}(n)$.

Success probability of \mathcal{E} . The initial call to \mathcal{E} on the root of \mathcal{T} returns “fail” if and only if the very first recursive call made by *every* invocation of \mathcal{E} in the call stack returns “fail”. That is, \mathcal{E} succeeds in outputting a tree of accepting transcripts when called on the root whenever its recursive call on the first child j of the root succeeds, which itself succeeds whenever its recursive call on the first child of j succeeds, and so forth. This probability is exactly the probability \mathcal{P} succeeds in convincing the verifier to accept, namely ϵ .

We still need to argue that the probability that conditioned on \mathcal{E} successfully outputting a tree, the probability that \mathcal{E} assigns any two edges in the graph the same challenge by \mathcal{E} is negligible. To argue this, let us assume that \mathcal{E} never runs for more than T time steps, for $T = p^{1/3}$. Here, p denotes the order of \mathbb{G} , and hence the size of the verifier’s challenge space. We can ensure this by having \mathcal{E} halt and output “fail” if it surpasses T time steps—by Markov’s inequality, since \mathcal{E} runs in expected time $\text{poly}(n)$, the probability \mathcal{E} exceeds T timesteps is at most $\text{poly}(n)/T$, which is negligible assuming p is superpolynomially large in n . Hence, after ensuring this assumption holds, the probability \mathcal{E} succeeds in outputting a tree of accepting transcripts is still at most ϵ minus a negligible quantity. If \mathcal{E} never runs for more than T timesteps, then it only can only generate at most T random challenges of the verifier over the course of its execution. The probability of a collision amongst these at most T challenges is bounded above by $T^2/p \leq 1/p^{1/3}$, which is negligible. We conclude as desired that the probability \mathcal{E} output a 3-transcript tree is at least ϵ minus a negligible quantity, which is at least $\epsilon/2$ if ϵ is non-negligible. \square

Step 2: Extracting a witness from any 3-transcript tree. Second, we must give a polynomial time algorithm that takes as input a 3-transcript tree for Protocol 12 and outputs a vector u such that $c_u = \sum_{i=1}^n u_i g_i$. The idea for how this is done is to iteratively compute a label u for each node in the tree, starting with the leaves and working layer-by-layer towards the root. For each node in the tree, the procedure will essentially

reconstruct the vector u that the prover must have “had in its head” at that stage of the protocol’s execution. That is, each node in the tree is associated with a vector of generators \mathbf{g}' and a commitment c , and the extractor will identify a vector u' such that $\langle u', \mathbf{g}' \rangle = c$.

Associating a generator vector and commitment with each node in the tree. For any node in the tree, we may associate with that node a generator vector and commitment in the natural way. That is, Protocol 12 is recursive, and each node in the tree at distance i from the root corresponds to a call to Protocol 12 at depth i of the call stack. As per Line 2, the verifier in each recursive call to Protocol 12 is aware of a generator vector and a commitment c (supposedly a commitment to some vector known to the prover, using the generator vector).

For example, the root of the tree is associated with $\mathbf{g} = \mathbf{g}_L \circ \mathbf{g}_R$ and commitment $c = c_u$ that is input to the original call to Protocol 12. If the root is associated with prover message (v_L, v_R) , then a child connected to the root by an edge of label α is associated with vector $\mathbf{g}' = \alpha^{-1} \mathbf{g}_L + \alpha \mathbf{g}_R$ and commitment $c' = c_u + \alpha^2 v_L + \alpha^{-2} v_R$, where v_L and v_R denote the prover messages associated with the edge. And so on down the tree.

Assigning a label to each node of the tree, starting with the leaves and working toward the root. Given a 3-transcript tree, begin by labelling each leaf with the prover’s final message in the protocol. Because every leaf transcript is accepting, if a leaf is assigned label u , generator \mathbf{g} , and commitment c , then we know that $g^u = c$.

Now assume by way of induction that, for each node at distance at most $\ell \geq 0$ from the leaves, if the node is associated with generator vector \mathbf{g} and commitment c , the label-assigning procedure has successfully assigned a label vector u to the node such that $\langle u, \mathbf{g} \rangle = c$. We explain how to extend the procedure to assign such labels to nodes at distance $\ell + 1$ from the leaves.

To this end, consider such a node j and let the associated generator vector be $\mathbf{g} = \mathbf{g}_L \circ \mathbf{g}_R$ and associated commitment be c . For $i = 1, 2, 3$, let \mathbf{g}_i and c_i denote the generator vector and commitment associated with j ’s i th child, u_i denote the label that has already been assigned to the i th child, and α_i denote the verifier challenge associated with the edge connecting j to its i th child. By construction of the generators and commitment associated with each node in the tree, for each i , the following two equations hold, relating the generators and commitment for node j to those of its children:

$$\mathbf{g}_i = \alpha_i \mathbf{g}_L + \alpha_i^{-1} \mathbf{g}_R \tag{13.12}$$

and

$$c_i = c + \alpha_i^2 v_L + \alpha_i^{-2} v_R. \tag{13.13}$$

Moreover, by the inductive hypothesis, the label-assigning algorithm has ensured that

$$\langle u_i, \mathbf{g}_i \rangle = c_i. \tag{13.14}$$

At an intuitive level, Equation (13.14) identifies a vector u_i “explaining” the commitment c_i of child i in terms of the generator vector \mathbf{g}_i , while Equations (13.12) and (13.13) relate c_i and \mathbf{g}_i to c and \mathbf{g} . We would like to put all of this information together to identify a vector u “explaining” c in terms of \mathbf{g} .

To this end, combining Equations (13.12)-(13.13), we conclude that

$$\langle u_i, \alpha_i \mathbf{g}_L + \alpha_i^{-1} \mathbf{g}_R \rangle = c + \alpha_i^2 v_L + \alpha_i^{-2} v_R,$$

and by applying the distributive law to the left hand side, we finally conclude that:

$$\langle \alpha_i u_i, \mathbf{g}_L \rangle + \langle \alpha_i^{-1} u_i, \mathbf{g}_R \rangle = c + \alpha_i^2 v_L + \alpha_i^{-2} v_R. \quad (13.15)$$

Equation (13.15) “almost” achieves our goal of identify a vector u such that $\langle u, \mathbf{g} \rangle = c$, in the sense that if the “cross terms” $\alpha_i^2 v_L + \alpha_i^{-2} v_R$ did not appear in Equation (13.15) for, say, $i = 1$, then the vector $u = \alpha_1^2 u_1 \circ \alpha_1^{-2} u_1$ would satisfy $\langle u, \mathbf{g} \rangle = c$. The point of deriving Equation (13.15) not only for $i = 1$, but also for $i = 2$ and $i = 3$ is that we can use the latter two equations to “cancel out the cross terms” from the right hand side of the equation for $i = 1$. Specifically, there exists some coefficients $\beta_1, \beta_2, \beta_3 \in \mathbb{F}_p$ such that

$$\sum_{i=1}^3 \beta_i \cdot (c + \alpha_i^2 v_L + \alpha_i^{-2} v_R) = c. \quad (13.16)$$

This follows from the fact that the following matrix is full rank, and hence has the vector $(1, 0, 0)$ in its row-span:

$$A = \begin{bmatrix} 1 & \alpha_1^2 & \alpha_1^{-2} \\ 1 & \alpha_2^2 & \alpha_2^{-2} \\ 1 & \alpha_3^2 & \alpha_3^{-2} \end{bmatrix}.$$

One way to see that A is invertible is to directly compute the determinant as $-\frac{(\alpha_1^2 - \alpha_2^2)(\alpha_1^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2)}{\alpha_1^2 \alpha_2^2 \alpha_3^2}$, which is clearly non-zero so long as α_1, α_2 , and α_3 are all distinct. Moreover, $(\beta_1, \beta_2, \beta_3)$ can be computed efficiently—in fact, it equals the first row of A^{-1} .

Equation (13.16) combined with Equation (13.15) implies that $u = \sum_{i=1}^3 (\beta_i \cdot \alpha_i \cdot u_i) \circ (\beta_i \cdot \alpha_i^{-1} \cdot u_i)$ satisfies $\langle u, \mathbf{g} \rangle = c$, where \circ denotes concatenation.

In this manner, labels can be assigned to each node in the tree, starting with the leaves and proceeding layer-by-layer towards the root. The label u assigned to the root satisfies $\langle u, \mathbf{g} \rangle = c_u$ as desired.

13.1.4.2 The Polynomial Commitment Scheme

The preceding section describes a (non-zero-knowledge) argument of knowledge of an opening $u \in \mathbb{F}_p^n$ of a generalized Pedersen commitment c_u , i.e., a u such that $\sum_{i=1}^n u_i \cdot g_i = c_u$. To obtain a (non-zero-knowledge) polynomial commitment scheme, we need to modify this argument of knowledge to establish not only that

$$\sum_{i=1}^n u_i \cdot g_i = c_u, \quad (13.17)$$

but also that

$$\sum_{i=1}^n u_i \cdot y_i = v \quad (13.18)$$

for some public vector $y \in \mathbb{F}_p^n$ and $v \in \mathbb{F}_p$ (recall from Section 13.1.2 that u will be the coefficient vector of the committed polynomial, y will be a vector derived from the point at which the verifier requests to evaluate the committed polynomial, and v will be the claimed evaluation of the polynomial).

The idea is that Equations (13.17) and (13.18) are of exactly the same form, namely they both involve computing the inner product of u with another vector (despite each g_i being a group element in \mathbb{G} , while each y_i is a field element in \mathbb{F}_p). So one can simply run two parallel invocations of Protocol 12, using the same verifier challenges in both, but with the second instance replacing the vector of group generators \mathbf{g} with the vector y , and the group element c_u with the field element v . See Figure 13 for a complete description of the protocol.

Protocol 13 Extending Protocol 12 to a polynomial commitment scheme.

- 1: Let \mathbb{G} be an *additive* cyclic group of prime order p over which the Discrete Logarithm relation is hard, with vector of generators $\mathbf{g} = (g_1, \dots, g_n)$. Let $y \in \mathbb{F}_p^n$ be a public vector and public value $v \in \mathbb{F}_p$.
 - 2: Input is $c_u = \text{Com}(u) := \sum_{i=1}^n u_i g_i$. Prover knows u , Verifier only knows c_u , \mathbf{g} , y , and v .
 - 3: If $n = 1$, the prover sends u to the verifier and the verifier checks that $u g_1 = c_u$ and that $u y_1 = v$.
 - 4: Otherwise, write $u = u_L \circ u_R$, $\mathbf{g} = \mathbf{g}_L \circ \mathbf{g}_R$, and $y = y_L \circ y_R$. Prover sends v_L, v_R claimed to equal $\langle u_L, g_R \rangle$ and $\langle u_R, g_L \rangle$, as well as v'_L, v'_R claimed to equal $\langle u_L, y_R \rangle$ and $\langle u_R, y_L \rangle$.
 - 5: Verifier responds with a randomly chosen $\alpha \in \mathbb{F}_p$.
 - 6: Recurse on commitment $c_{u'} := c_u + \alpha^2 v_L + \alpha^{-2} v_R$ to vector $u' = \alpha u_L + \alpha^{-1} u_R$ of length $n/2$, using the vector of group generators $\mathbf{g}' := \alpha^{-1} \mathbf{g}_L + \alpha \mathbf{g}_R$, and using public vector $y' := \alpha^{-1} y_L + \alpha y_R$ and public value $v' = v + \alpha^2 v_L + \alpha^{-2} v_R$.
-

Sketch of how to achieve zero-knowledge. To render Protocol 13 zero-knowledge, one can apply commit-and-prove style techniques. This means that in every round, the prover does not send v'_L and v'_R to the verifier in the clear, but rather sends Pedersen commitments to these quantities (if one wants perfect rather than computational zero-knowledge, then a blinding factor h^z for randomly chosen z should be included in the Pedersen commitments as per Protocol 5; likewise, the group elements v_L and v_R sent in each round should be blinded as well). At the very end of the protocol, the prover proves in zero-knowledge that the committed values sent over the course of the $\log_2 n$ rounds of the protocol would have passed the check performed by the verifier in the final round of Protocol 13 (Line 3).¹³⁴ The protocol is public coin, and hence can be rendered non-interactive using the Fiat-Shamir transformation.

13.1.5 Dory: Reducing Verifier Time To Logarithmic

Recall that the polynomial commitment scheme of the previous section achieved constant commitment size, and evaluation proofs consisting of $O(\log n)$ group elements, but both the prover and verifier had to perform $\Theta(n)$ exponentiations in the group \mathbb{G} . Lee [Lee20] showed how to reduce the verifier’s runtime to $O(\log n)$ group exponentiations, following a one-time setup phase costing $O(n)$ group exponentiations. Lee naming the resulting commit scheme Dory.

More precisely, the setup phase in Dory produces a logarithmic-sized “verification key” derived from the length- n public vector of group generators \mathbf{g} such that any party with the verification key can implement the verifier’s checks in Protocol 13 in only $O(\log n)$ rather than $O(n)$ group exponentiations. One can think of the verification key as a small “summary” of the public vector \mathbf{g} that suffices to implement the verifier’s check in $O(\log n)$ time, in the sense that once the verifier computes the verification key, it can forget the actual generators that the key summarizes.

Note that, unlike the polynomial commitments of the next section (Section 13.2), this pre-processing phase is *not* what is called a *trusted setup*, which refers to a pre-processing phase produces “toxic waste” (also called a trapdoor) such that any party with the trapdoor can break binding of the polynomial commitment scheme. That is, while the setup phase in Dory produces a structured verification key (meaning the key does not consist of randomly chosen group elements), there is no trapdoor, and anyone willing to invest the computational effort can derive the key. Protocols such as Dory that do not required a trusted setup are often called *transparent*.

We limit ourselves to a *very* rough sketch of how Dory works. Recall that a generalized Pedersen

¹³⁴Bulletproofs [BBB⁺18] contains an optimization that reduces the number of commitments sent by the prover in each round from 4 to 2, by effectively compressing the two commitments v_L and v'_L into a single commitment, and similarly for v_R and v'_R .

commitment c_u is a commitment to an n -dimensional vector u with elements from \mathbb{F}_p , and is obtained by taking the “inner product” of u with a generator vector $\mathbf{g} \in \mathbb{G}^n$, where \mathbb{G} is a cryptographic group of order p . The commitment is compressing, in that it consists of only one group element (while the committed vector u has dimension n). Dory makes use of compressing commitments not only to vectors of elements in \mathbb{F}_p , but also to vectors of group elements (such as the generator vector \mathbf{g} itself). Such commitments were given by Abe et al. [AFG⁺10]. For purposes of this section, it will not matter much how these “commitments to vectors of group elements” are implemented, but a few sentences about them are merited. They make use of a notion called pairing-based cryptography, which we discuss in detail in the next section (Section 13.2.1). Pairings can be used to define a notion of an inner product between two vectors of group elements, and Abe et al. [AFG⁺10] observe that under suitable cryptographic assumptions, the inner product of \mathbf{g} and \mathbf{h} represents a compressing commitment to vector $\mathbf{g} \in \mathbb{G}^n$ using public generator vector \mathbf{h} .

Let us focus on the simplified setting of Protocol 12, which demonstrates knowledge of an opening of a generalized Pedersen commitment. The verifier’s runtime bottleneck in Protocol 12 is as follows. In the first round of Protocol 12, the prover and verifier need to update the public generator vector $\mathbf{g} = \mathbf{g}_L \circ \mathbf{g}_R$ to $\mathbf{g}' = \alpha^{-1}\mathbf{g}_L + \alpha\mathbf{g}_R$, where α is the random challenge chosen by the verifier in that round. This takes time proportional to the length of \mathbf{g} .

To avoid this expense for the verifier, the idea of Dory is that the pre-processing phase produces two constant-sized commitments, one to \mathbf{g}_L and one to \mathbf{g}_R . Both commitments are computed using a random generator vector \mathbf{h} of length $n/2$ (note that since \mathbf{h} is only a constant factor smaller than \mathbf{g} itself, we also do not want the verifier to have to read all of \mathbf{h} during the online phase of the protocol, and hence the pre-processing phase of Dory will also have to produce commitments to the first and second half of \mathbf{h} itself. For this high-level summary, let us ignore this issue and assume the verifier does know \mathbf{h} in the online phase of Dory).

Then, in the first round of the online phase, rather than the verifier explicitly computing \mathbf{g}' entry-by-entry, the verifier will instead simply compute a commitment to \mathbf{g}' (under generator vector \mathbf{h}) using the homomorphism property of the commitment scheme. This only requires the verifier to perform a constant number of group exponentiations. At this point, the verifier knows (ostensibly) a commitment c_u to a length $n/2$ vector u' under \mathbf{g}' , i.e.,

$$c_{u'} = \langle u', \mathbf{g}' \rangle, \tag{13.19}$$

and a commitment $c_{\mathbf{g}'}$ to \mathbf{g}' under \mathbf{h} , i.e.,

$$c_{\mathbf{g}'} = \langle \mathbf{h}, \mathbf{g}' \rangle. \tag{13.20}$$

While the verifier does *not* know either u' or \mathbf{g}' , it is sufficient for the prover to establish knowledge of u' and \mathbf{g}' satisfying Equations (13.19) and (13.20). Hence, roughly speaking Dory has reduced the task of proving knowledge of a length- n vector u satisfying $\langle u, \mathbf{g} \rangle = c_u$ to two instances of the same type of task on vectors of half the length. One can “combine” these two instances instead a single instance using a technique similar to Protocol 12 itself. The prover and verifier can then apply Dory recursively to that single instance. While this outline conveys the general approach of Dory, it elides many details.

Combining techniques. The protocol of Section 13.1.3 that leveraged the polynomial commitment scheme of Section 13.1.2 as a subroutine can replace the subroutine with any extractable additively-homomorphic vector-commitment scheme supporting inner product queries, including Bulletproofs (Section 13.1.4) or Dory (this section). If combined with Bulletproofs, the resulting scheme reduces the public parameter size of Bulletproofs from D to $\Theta(\sqrt{D})$, maintains an evaluation-proof size of $O(\log D)$ group elements, and

reduces the number of group exponentiations the verifier has to perform at the end of the protocol from D to $\Theta(\sqrt{D})$. The downside relative to vanilla Bulletproofs is that the size of the commitment increases from one group element to $\Theta(\sqrt{D})$ group elements. If combined with Dory, the resulting scheme does not asymptotically reduce any costs relative to Dory alone, but does reduce constant factors in the prover’s runtime [Lee20, SL20].

In Section 13.5, we briefly describe additional polynomial commitment schemes inspired by similar techniques, but based on cryptographic assumptions other than hardness of the discrete logarithm problem.

13.2 Polynomial Commitments from Pairings and a Trusted Setup

In this section, we explain how to use a cryptographic primitive called pairings (also referred to as bilinear maps) to give a simple and concretely efficient polynomial commitment scheme. A major benefit of this scheme is that commitments and openings consist of only a constant number of group elements. A downside is that it requires a *structured reference string* (SRS) that is as long as the number of coefficients in the polynomial being committed to. This string must be generated in a specified manner and made available to any party that wishes to commit to a polynomial. The generation procedure produces “toxic waste” (also called a trapdoor) that must be discarded. That is, whatever party generates the reference string knows a piece of information that would let the party break the binding property of the polynomial commitment scheme, and thereby destroy soundness of any argument system that uses the commitment scheme. The generation of such an SRS is also called a *trusted setup*. The techniques of this section date to work of Kate, Zaverucha, and Goldberg [KZG10], and hence are often referred to as KZG commitments.

13.2.1 Cryptographic Background

The Decisional Diffie-Helman Assumption The Decisional Diffie-Helman (DDH) assumption in a cyclic group \mathbb{G} with generator g states that, given g^a and g^b for a, b chosen uniformly and independently from $|\mathbb{G}|$, the value g^{ab} is computationally indistinguishable from a random group element. Formally, the assumption is that the following two distributions cannot be distinguished, except for negligible advantage over random guessing, by any efficient algorithm:

- (g^a, g^b, g^{ab}) where a and b are chosen uniformly at random from $|\mathbb{G}|$.
- (g^a, g^b, g^c) where a and b and c are chosen uniformly at random from $|\mathbb{G}|$.

If one could compute discrete logarithms efficiently in \mathbb{G} , then one could break the DDH assumption in that group: given as input a triple of group elements (g_1, g_2, g_3) , one could compute the discrete logarithms a, b, c of g_1, g_2, g_3 in base g , and check whether $c = a \cdot b$, outputting “yes” if so. This algorithm would always output yes under draws from the first distribution above, and output yes with probability just $1/|\mathbb{G}|$ under draws from the second distribution.

Hence, the DDH assumption is a stronger assumption than hardness of the Discrete Logarithm problem. In fact, there are groups in which the DDH assumption is false, yet the discrete logarithm problem is nonetheless believed to be hard.

A close relative of DDH is the *computational Diffie-Helman* (CDH) assumption, which states that given g^a and g^b , no efficient algorithm can compute g^{ab} . CDH is a weaker assumption than DDH in the sense that if one can compute g^{ab} given g^a and g^b , then one can also solve the DDH problem of distinguishing (g^a, g^b, g^{ab}) for (g^a, g^b, g^c) for random group elements $a, b, c \in \mathbb{G}$.

Pairing-friendly Groups and Bilinear Maps. Let \mathbb{G} and \mathbb{G}_t be two cyclic groups of the same order. A map $e: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_t$ is said to be bilinear if for all $u, v \in \mathbb{G}$ and $a, b \in \{0, \dots, |\mathbb{G}| - 1\}$, $e(u^a, v^b) = e(u, v)^{ab}$.¹³⁵ If a bilinear map e is also non-degenerate (meaning, it does not map all pairs in $\mathbb{G} \times \mathbb{G}$ to the identity element $1_{\mathbb{G}_t}$) and e is efficiently computable, then e is called a pairing. This terminology refers to the fact that e associates each pair of elements in \mathbb{G} to an element of \mathbb{G}_t .

Note that any two cyclic groups \mathbb{G} and \mathbb{G}_t of the same order are in fact isomorphic, meaning there is a bijective mapping $\pi: \mathbb{G} \mapsto \mathbb{G}_t$ that preserves group operations, i.e., $\pi(a \cdot b) = \pi(a) \cdot \pi(b)$ for all $a, b \in \mathbb{G}$. But just because \mathbb{G} and \mathbb{G}_t are isomorphic does not mean they are equivalent from a computational perspective; elements of \mathbb{G} and \mathbb{G}_t and the respective group operations can be represented and computed in very different ways.

Not all cyclic groups \mathbb{G} for which the discrete logarithm problem is believed to be hard are “pairing-friendly”, i.e., come with a bilinear map e mapping $\mathbb{G} \times \mathbb{G}$ to \mathbb{G}_t . For example, as elaborated upon shortly, the popular Curve25519, which is believed to yield an elliptic curve group in which the discrete logarithm problem is hard, is not pairing-friendly. As a result, group operations of pairing-friendly elliptic curves tend to be concretely slower than preferred groups that need not be pairing-friendly.

In more detail, in practice, if \mathbb{G} is an elliptic curve group defined over field \mathbb{F}_p , then \mathbb{G}_t is typically a multiplicative subgroup of an extension field \mathbb{F}_{p^k} for some positive integer k (recall from Section 2.1.5 that \mathbb{F}_{p^k} denotes the finite field of size p^k). That is, \mathbb{G}_t consists of (a subgroup of the) nonzero elements of \mathbb{F}_{p^k} , with the group operation being field multiplication. As the multiplicative subgroup of \mathbb{F}_{p^k} has size $p^k - 1$, and the order $|\mathbb{H}|$ of any subgroup \mathbb{H} of a group \mathbb{G}' divides $|\mathbb{G}'|$, k is chosen to be the smallest integer such that $|\mathbb{G}|$ divides $p^k - 1$; this value of k is called the *embedding degree* of \mathbb{G} . Hence, to efficiently implement pairings in this manner, \mathbb{G} must have low embedding degree. Unfortunately, popular groups in which the Discrete Logarithm problem is believed intractable, such as Curve25519, have enormous embedding degree. This is why arithmetic in pairing-friendly groups tends to be concretely slower than in non-pairing-friendly groups. At the time of writing, a popular pairing-friendly curve for use in SNARKs is called BLS12-381, which has embedding degree 12 and targets roughly 120 bits of security.¹³⁶

Note that in any group \mathbb{G} equipped with a pairing, the Decisional Diffie-Hellman assumption does not hold. This is because one can distinguish triples (g_1, g_2, g_3) of the form $(g_1 = g^a, g_2 = g^b, g_3 = g^{ab})$ from $(g_1 = g^a, g_2 = g^b, g_3 = g^c)$ for randomly chosen $c \in |\mathbb{G}|$ by checking whether $e(g, g_3) = e(g_1, g_2)$. In the case where $g_3 = g^{ab}$, this check will always pass by bilinearity of e , while if e is non-degenerate, this check will fail with noticeable probability if g_3 is a random group element in \mathbb{G} . Nonetheless, in pairing-friendly groups it is often assumed that the *computational* Diffie-Hellman assumption holds.

Intuition for Why Bilinear Maps are Useful. Recall that an additively homomorphic commitment scheme such as Pedersen commitments allows any party to perform addition “underneath commitments”. That is, despite the fact that the commitments perfectly hide the value that is committed, it is possible for anyone to take two commitments c_1, c_2 to values m_1, m_2 , and compute a commitment c_3 to $m_1 + m_2$, despite not actually knowing anything about m_1 or m_2 . However, Pedersen commitments are not multiplicatively homomorphic: while we gave an efficient interactive protocol for a prover (that knows how to open c_1 and c_2)

¹³⁵In general, the domain of a bilinear map might consist of pairs of elements from two *different* cyclic groups $\mathbb{G}_1, \mathbb{G}_2$ of the same order as \mathbb{G}_t , rather than pairs of elements from the same cyclic group \mathbb{G} . In the general case that $\mathbb{G}_1 \neq \mathbb{G}_2$, the pairing is said to be *asymmetric*, while the case that $\mathbb{G}_1 = \mathbb{G}_2$ is called *symmetric*. Asymmetric pairings can be much more efficient in practice than symmetric pairings. For simplicity in this manuscript we will only consider the symmetric case in which $\mathbb{G}_1 = \mathbb{G}_2$.

¹³⁶See <https://electriccoin.co/blog/new-snark-curve/> and <https://hackmd.io/@benjaminion/bls12-381> for discussion of BLS12-381.

to prove that c_3 commits to $m_1 \cdot m_2$, it is not possible for a party that does not know m_1 or m_2 to compute a commitment to $m_1 \cdot m_2$ on its own.

Bilinear maps effectively convey the power of multiplicative-homomorphism, but only for *one* multiplication operation. To be more concrete, let us think of a group element $g^{m_i} \in \mathbb{G}$ as a commitment to m_i (the commitment is computationally hiding if the discrete logarithm problem is hard in \mathbb{G} , meaning it is hard to determine m_i from g^{m_i}). Then bilinear maps allow any party, given commitments c_1, c_2, c_3 to check whether the values m_1, m_2, m_3 inside the commitments satisfy $m_3 = m_1 \cdot m_2$. This is because by bilinearity of the map $e: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_t$, $e(g^{m_1}, g^{m_2}) = e(g^{m_3}, g)$ if and only if $m_3 = m_1 \cdot m_2$.

It turns out that the power to perform a single “multiplication check” of committed values is enough to obtain a polynomial commitment scheme. This is because Lemma 8.3 implies that for any degree- D univariate polynomial p , the assertion “ $p(z) = v$ ” is equivalent to the assertion that there exists a polynomial w of degree at most $D - 1$ such that

$$p(X) - v = w(X) \cdot (X - z). \quad (13.21)$$

Equation (13.21) can be probabilistically verified by evaluating the two polynomials on the left hand side and right hand side at a randomly chosen point τ . This intuitively means that the committer can commit to p by sending a commitment c_3 to $m_3 := p(\tau)$, and then convince a verifier that Equation (13.21) holds by sending a commitment c_2 to $m_2 := w(\tau)$. If the verifier can compute a commitment c_1 to $m_1 := \tau - z$ on its own, then the verifier can use the bilinear map to check that indeed the $m_3 = m_1 \cdot m_2$ (i.e., Equation (13.21) holds at input τ). This entire approach assumes that the committer does not know τ , since if it did, it could choose the polynomial w so that Equation (13.21) does not hold as an equality of polynomials, but does hold at τ .

The following section makes the above high-level outline formal.

13.2.2 Polynomial Commitment for Univariate Polynomials from Pairings

A binding scheme. Let e be a bilinear map pairing groups \mathbb{G}, \mathbb{G}_t of prime order p , and $g \in \mathbb{G}$ be a generator, and D be an upper bound on the degree of the polynomials we would like to support commitments to. The structured reference string consists of encodings in \mathbb{G} of all powers of a random nonzero field element $\tau \in \mathbb{F}_p$. That is, τ is an integer chosen at random from $\{1, \dots, p - 1\}$, and the SRS equals $(g, g^\tau, g^{\tau^2}, \dots, g^{\tau^D})$. The value τ is toxic waste that must be discarded because it can be used to destroy binding.

To commit to a polynomial q over \mathbb{F}_p , the committer sends a value c claimed to equal $g^{q(\tau)}$. Note that while the committer does not know τ , it is still able to compute $g^{q(\tau)}$ using the SRS and additive homomorphism: if $q(Z) = \sum_{i=0}^D c_i Z^i$, then $g^{q(\tau)} = \prod_{i=0}^D (g^{\tau^i})^{c_i}$, which can be computed given the values g^{τ^i} for all $i = 0, \dots, D$ even without knowing τ .¹³⁷

To open the commitment at input $z \in \{0, \dots, p - 1\}$ to some value v , i.e., to prove that $q(z) = v$, the committer computes a “witness polynomial”

$$w(X) := (q(X) - v)/(X - z),$$

¹³⁷As in Section 13.1.4, one can think of g^{τ^i} as a Pedersen commitment to τ^i (Protocol 5), but without the blinding factor h^z . This yields a commitment that is perfectly binding but only computationally hiding (since g^{τ^i} information-theoretically specifies τ^i , but deriving τ^i from g^{τ^i} requires computing the discrete logarithm of g^{τ^i} to base g). The ability of the committer to compute $g^{q(\tau)}$ from the SRS without knowing τ follows from the fact that this modified Pedersen commitment is additively homomorphic.

and sends a value y claimed to equal $g^{w(\tau)}$ to the verifier. Again, since w has degree at most D , $g^{w(\tau)}$ can be computed from the SRS despite the fact that the prover does not know τ . The verifier checks that

$$e(c \cdot g^{-v}, g) = e(y, g^\tau \cdot g^{-z}).$$

Note that this requires the verifier to know c, v, y, z , and g^τ . The first three values are provided by the prover, the opening query z is determined by the verifier itself, and g^τ is an entry of the SRS (note that g^τ and g are the only entries of the SRS needed for verification. For this reason, some works refer to the entire SRS as the *proving key* and (g, g^τ) as the *verification key*, and think of the verifier as only downloading the verification key, not the entire proving key).

Analysis of correctness and binding. Correctness is easy to establish: if $c = g^{q(\tau)}$, $y = g^{w(\tau)}$ then

$$e(c \cdot g^{-v}, g) = e(g^{q(\tau)-v}, g) = e(g^{w(\tau) \cdot (\tau-z)}, g) = e(g^{w(\tau)}, g^{\tau-z}) = e(y, g^\tau \cdot g^{-z}).$$

Here, the first inequality holds because $c = g^{q(\tau)}$, the second holds by definition of w as $(q(X) - v)/(X - z)$, the third holds by bilinearity of e , and the fourth holds because $y = g^{w(\tau)}$.

The intuition for why binding holds is that, if $q(z) \neq v$, then $w(X) = (q(X) - v)/(X - z)$ is not a polynomial but rather a rational function, and hence it is not possible for the prover to evaluate $w(\tau)$ using the SRS (as this would require computing $g^{1/(\tau-z)}$, by which we mean the group element $h \in \mathbb{G}$ such that $h^{(\tau-z)} = g$, and $g^{1/(\tau-z)}$ is not part of the SRS). Specifically, binding follows from a cryptographic assumption that is essentially tailor-made to ensure security of this scheme.

The assumption (called the *D-strong Diffie-Hellman (SDH) assumption*) asserts that, given the SRS that consists of all the generator g raised to all powers of τ up to power- D , there is no efficient algorithm \mathcal{A} that on input z can output $g^{1/(\tau-z)}$ except with negligible probability. Note that this assumption implies that the Discrete Logarithm problem is hard in \mathbb{G} , because if discrete logarithms are easy to compute, then τ can be efficiently computed from g^τ , and given τ and z it is easy to compute $g^{1/(\tau-z)}$. Indeed, this can be done by computing the multiplicative inverse ℓ modulo p of $\tau - z$ using the Extended Euclidean algorithm. Since \mathbb{G} has order p , and hence $g^{ip} = 1_{\mathbb{G}}$ for all integers i , $g^\ell = g^{1/(\tau-z)}$.

Formally, one must show that if one can open a commitment c at point $z \neq \tau$ to different values v, v' , then one can efficiently compute $g^{1/(\tau-z)}$, violating the SDH assumption. To open c to values v and v' the committer must identify values $y, y' \in \mathbb{G}$ such that:

$$e(c \cdot g^{-v}, g) = e(y, g^\tau \cdot g^{-z})$$

and

$$e(c \cdot g^{-v'}, g) = e(y', g^\tau \cdot g^{-z}).$$

For simplicity, let us write $c = g^{r_1}$, $y = g^{r_2}$, and $y' = g^{r_3}$ (although the committer may not know r_1, r_2 , or r_3). By bilinearity of e , these two equations imply that

$$g^{r_1} \cdot g^{-v} = g^{r_2 \cdot (\tau-z)}$$

and

$$g^{r_1} \cdot g^{-v'} = g^{r_3 \cdot (\tau-z)}.$$

Together, these two equations imply that:

$$g^{v'-v} = g^{(r_3-r_2)(\tau-z)}.$$

In other words,

$$\left((y' \cdot y^{-1})^{1/(v'-v)} \right)^{(\tau-z)} = g. \quad (13.22)$$

Here, $(y' \cdot y^{-1})^{1/(v'-v)}$ denotes the value obtained raising the group element $y' \cdot y^{-1} \in \mathbb{G}$ to the power x , where x is the multiplicative inverse of $v' - v$ modulo p ; note that x can be computed efficiently (in time $\tilde{O}(\log p)$) via the Extended Euclidean algorithm. Equation (13.22) states that $\left((y' \cdot y^{-1})^{1/(v'-v)} \right)$ equals $g^{1/(\tau-z)}$. Since this value can be computed efficiently given v, v', y, y' provided by the committer, the committer must have broken the SDH assumption.

An extractable scheme. To make this polynomial commitment scheme extractable rather than simply binding, it must be modified and/or additional cryptographic assumptions are required (see for example the discussion of the Generic Group Model later in this section).

Here is one method of achieving extractability that does require modifying the scheme, as well as an additional cryptographic assumption. Recall that \mathbb{G} is a cyclic group of order p with public generator g . In the modified scheme, the SRS doubles in size. Specifically, for τ and α chosen at random from \mathbb{F}_p , the modified SRS consists of the pairs

$$\{(g^\tau, g^{\alpha\tau}), (g^{\tau^2}, g^{\alpha\tau^2}), \dots, (g^{\tau^D}, g^{\alpha\tau^D})\}.$$

That is, the SRS consists not only of powers of τ in the exponent of g , but also the same quantities raised to the power α . Note that neither τ nor α are included in the SRS—they are “toxic waste” that must be discarded after the SRS is generated, as any party that knows these quantities can break extractability or binding of the polynomial commitment scheme.

The Power Knowledge of Exponent (PKoE) assumption [Gro10a] assumes that for any polynomial time algorithm given access to the SRS, whenever the algorithm outputs any two group elements $g_1, g_2 \in \mathbb{G}$ such that $g_2 = g_1^\alpha$, it is possible to efficiently extract from the algorithm a sequence of coefficients c_1, \dots, c_D that “explain” g_2 , in the sense that $g_2 = \prod_{i=1}^D g^{c_i \tau^i}$. The idea of the assumption is that, given access to the SRS, it is easy to compute a pair (g_1, g_1^α) in the following manner: let g_1 equal any product of quantities in the first half of the SRS raised to constant powers, e.g.,

$$g_1 := \prod_{i=1}^D g^{c_i \tau^i},$$

and let g_2 be the result of applying the same operations to the second half of the SRS, i.e., $g_2 := \prod_{i=1}^D g^{c_i \alpha \tau^i}$. The PKoE essentially assumes that this is the *only* way that an efficient party is capable of computing two group elements with this relationship to each other.

Whereas in the original commitment scheme (that was binding but not necessarily extractable), the commitment to polynomial q was $g^{p(\tau)}$ computed by the committer using the SRS, in the modified scheme the commitment is the pair $(g^{p(\tau)}, g^{\alpha p(\tau)})$, which the committer can compute using the modified SRS. To open a commitment $c = (U, V)$ at $z \in \mathbb{F}_p$ to value y , the committer computes the polynomial $w(X) := (q(X) - v)/(X - z)$ and sends a value y claimed to equal $w(\tau)$ exactly as in the original scheme. The verifier checks not only that

$$e(U \cdot g^{-v}, g) = e(y, g^\tau \cdot g^{-z}),$$

but also that $e(U, g^\alpha) = e(V, g)$.

Completeness for the first check holds exactly as in the unmodified scheme. Completeness for the verifier’s second check holds because if U and V are provided honestly then $V = U^\alpha$ (despite the fact that neither the prover nor the verifier know α), and hence by bilinearity of e , $e(U, g^\alpha) = e(V, g)$.

To prove that the modified scheme is extractable, we use the extractor whose existence is asserted by the PKoE assumption to construct an extractor \mathcal{E} for the polynomial commitment scheme. Specifically, the second check made by the verifier during opening ensures that $V = U^\alpha$ (despite the fact that the verifier does not know what α is). The PKoE assumption therefore asserts the existence of an efficient extraction procedure that, given access to the committer, outputs quantities c_1, \dots, c_D such that $U = \prod_{i=1}^D g^{c_i \tau^i}$. We define the polynomial extractor \mathcal{E} to output the polynomial $s(X) = \sum_{i=1}^D c_i X^i$. Clearly, $s(\tau) = U$, so (U, V) is indeed a commitment to the polynomial s . Moreover, if $s(z) \neq v$ for a (point, evaluation)-pair for which the committer is able to open $c = (U, V)$ to value v at point z , then the committer can break binding of the original unmodified commitment scheme.¹³⁸ This is because the committer is able to open c at point z to both v and $s(z)$ (the committer can open to $s(z)$ by letting $w'(X) = (s(X) - v)/(X - z)$ and sending $g^{w'(\tau)}$ during the opening procedure; this quantity will pass the verifier’s first check by the completeness analysis of the unmodified commitment scheme).

Discussion of the PKoE Assumption. The PKoE assumption is qualitatively different from all other cryptographic assumptions that we have discussed thus far in this manuscript, including the DDH and CDH assumptions, the discrete logarithm assumption, and the existence of collision-resistant families of hash functions. Specifically, all of these other assumptions satisfy a property called *falsifiability*. Falsifiability is a technical notion formalized by Naor [Nao03]: a cryptographic assumption is said to be falsifiable if it can be captured by defining an interactive game between a polynomial-time challenger and an adversary, at the conclusion of which the challenger can decide in polynomial-time whether the adversary won the game. A falsifiable assumption must be of the form “every efficient adversary has a negligible probability of winning the game”.

For example, the assumption that a hash family is collision-resistant can be modeled by having the challenger send the adversary a hash function h chosen at random from the family, and challenging the adversary to find a collision, i.e., two distinct strings x, y such that $h(x) = h(y)$. Clearly, the challenger can efficiently check whether the adversary won the game, by evaluating h at x and y and confirming that indeed $h(x) = h(y)$. In contrast, a knowledge-of-exponent assumption such as PKoE is not falsifiable: if the adversary computes a pair (g_1, g_1^α) , it is not clear how the challenger could determine whether the adversary broke the assumption. That is, since the challenger does not have access to the internal workings of the adversary, it is not clear how the challenger could determine whether or not the adversary computed (g_1, g_1^α) without the adversary “knowing in its own head” coefficients c_1, \dots, c_D such that $g_1 = \prod_{i=1}^D g^{c_i \tau^i}$.

Theoretical cryptographers generally prefer falsifiable assumptions because they seem easier to reason about and are arguably more concrete, as there is an efficient process to check whether an adversarial strategy falsifies the assumption. That said, not all falsifiable assumptions are “superior” to all non-falsifiable ones: indeed, some falsifiable assumptions proposed in the research literature have turned out to be false! And cryptographers certainly do believe that the PKoE assumption holds in many groups.

We have presented some succinct *interactive* arguments for circuit satisfiability in this manuscript that are based on falsifiable assumptions (e.g., the 4-message argument obtained by combining PCPs

¹³⁸The SRS of the modified commitment scheme contains additional group elements $g^{\alpha\tau}, g^{\alpha\tau^2}, \dots, g^{\alpha\tau^D}$ for a random $\alpha \in \mathbb{F}$. Any adversary \mathcal{A} for the SDH assumption that is given access to the original SRS can efficiently simulate these extra group elements itself by picking α at random and raising every element of the unmodified SRS to the power α . Hence, these extra group elements do not give the SDH adversary any extra power.

with Merkle trees from Section 8.2). But none of the *non-interactive* succinct arguments of knowledge (SNARKs) for circuit satisfiability that we present are based on falsifiable assumptions; they are either based on knowledge-of-exponent assumptions such as PKoE, or they are sound in the random oracle model.¹³⁹ This is because it is not known how to base a SNARK for circuit satisfiability on a falsifiable assumption, and indeed some barriers to achieving this are known [GW11].

In summary, while assumptions like PKoE are slightly controversial in the theoretical cryptography community, many researchers and practitioners are nonetheless confident in their veracity. It is perhaps reasonable to expect that any given deployed SNARK is more likely to be broken for mundane reasons such as unnoticed flaws in the security proofs or bugs in the implementation, than because the PKoE assumption turns out to be false in the group that used by the SNARK.

Generic Group Model and Algebraic Group Model. The unmodified polynomial commitment scheme covered in this section is also known to be extractable in the so-called *Generic Group model* (GGM) (in fact, in a variant of this model called the *Algebraic Group model* (AGM) [FKL18]). The Generic Group model is similar in spirit to the random oracle model (see Section 4.7.1). Recall that the random oracle model models cryptographic hash functions as truly random functions. In contrast, “real-world” implementations of protocols in the random oracle model must instantiate the random oracle with concrete hash functions, and real-world attackers trying to “break” the protocol can try to exploit properties of the concrete hash function. Accordingly, the random oracle model only captures “attacks” that do not exploit any structure in the concrete hash functions. The rationale for why this is reasonable is that real-world cryptographic hash functions are designed to (hopefully) “look random to efficient adversaries”; hence we generally do not know real-world attacks that exploit structure in concrete hash functions (though contrived protocols are known for which no real-world instantiation of the protocol with a concrete hash function is secure).

Similarly, the GGM considers adversaries that are only given access to cryptographic groups \mathbb{G}, \mathbb{G}_t via an oracle that computes the group multiplication operation (the pairing operation $e: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_t$ is modeled as an additional oracle). In the real-world, attackers are actually given efficient encodings of the group (e.g., an explicit representation of the elliptic curve or finite field used to define the groups), but we generally do not know of attacks on real-world protocols that exploit the efficient encodings. The AGM is a model that lies in between the GGM and the standard model in which adversaries are given efficient encodings of the group. The AGM has a similar flavor to knowledge-of-exponent assumptions like PKoE, in that it assumes whenever an efficient algorithm \mathcal{A} outputs a group element $g \in \mathbb{G}$, it also outputs an “explanation” of g as a combination of group elements $L = (L_1, \dots, L_t)$ that were previously given to \mathcal{A} , i.e., numbers c_1, \dots, c_t such that $g = \prod_{i=1}^t L_i^{c_i}$. Any attacker operating in the GGM can also be implemented in the AGM, justifying the assertion that the AGM lies in between the standard model and the GGM.

13.2.3 Extension to Multilinear Polynomials

The previous section gave a polynomial commitment scheme based on pairings, for univariate polynomials over the field \mathbb{F}_p where p is the order of the groups involved in the pairing. In this section, we wish to give a similar commitment scheme for *multilinear* polynomials q over \mathbb{F}_p , proposed by Papamanthou, Shi, and Tamassia [PST13]. Let ℓ denote the number of variables of q , so $q: \mathbb{F}_p^\ell \rightarrow \mathbb{F}_p$. In applications of multilinear polynomial commitment schemes (namely, to transforming IPs and MIPs to succinct arguments for circuit satisfiability), it is convenient to work with polynomials specified over the Lagrange basis (see Lemma 3.8

¹³⁹We did explain a succinct non-interactive argument for circuit *evaluation* based on falsifiable assumptions in Section 4.7.2, by instantiating the Fiat-Shamir transformation of the GKR protocol with using a correlation-intractable hash family.

for a definition of the Lagrange basis polynomials), so we present the commitment scheme in this setting, though the scheme works just as well over any basis of multilinear polynomials.

The structured reference string (SRS) now consists of encodings in \mathbb{G} of all powers of all Lagrange basis polynomials evaluated at a randomly chosen input $r \in \mathbb{F}^\ell$. That is, if $\chi_1, \dots, \chi_{2^\ell}$ denotes an enumeration of the 2^ℓ Lagrange basis polynomials, the SRS equals $(g^{\chi_1(r)}, \dots, g^{\chi_{2^\ell}(r)})$. Once again, the toxic waste that must be discarded because it can be used to destroy binding is the value r .

As in the univariate commitment scheme, to commit to a multilinear polynomial q over \mathbb{F}_p , the committer sends a value c claimed to equal $g^{q(r)}$. Note that while the committer does not know r , it is still able to compute $g^{q(r)}$ using the SRS: if $q(X) = \sum_{i=0}^{2^\ell} c_i \chi_i(X)$, then $g^{q(r)} = \prod_{i=0}^{2^\ell} (g^{\chi_i(r)})^{c_i}$, which can be computed given the values $g^{\chi_i(r)}$ for all $i = 0, \dots, 2^\ell$ even without knowing r .

To open the commitment at input $z \in \mathbb{F}_p^\ell$ to some value v , i.e., to prove that $q(z) = v$, the committer computes a series of ℓ “witness polynomials” w_1, \dots, w_ℓ , defined in the following fact.

Fact 13.2 (Papamanthou, Shi, and Tamassia [PST13]). *For any fixed $z = (z_1, \dots, z_\ell) \in \mathbb{F}_p^\ell$ and any multilinear polynomial q , $q(z) = v$ if and only if there is a unique set of ℓ multilinear polynomials w_1, \dots, w_ℓ such that*

$$q(X) - v = \sum_{i=1}^{\ell} (X_i - z_i) w_i(X). \quad (13.23)$$

Proof. If $q(X) - v$ can be expressed as the right hand side of Equation (13.23), then clearly $q(z) - v = 0$, and hence $q(z) = v$.

On the other hand, suppose that $q(z) = v$. Then by dividing the polynomial $q(X) - v$ by the polynomial $(X_1 - z_1)$, we can identify multilinear polynomials w_1 and s_1 such that

$$q(X) - v = (X_1 - z_1) \cdot w_1(X_1, X_2, \dots, X_\ell) + s_1(X_2, X_3, \dots, X_\ell),$$

where $s_1(X_2, X_3, \dots, X_\ell)$ is the remainder term, and does not depend on variable X_1 . Iterating this process, we can divide s_1 by the polynomial $(X_2 - z_2)$ to write

$$q(X) - v = (X_1 - z_1) \cdot w_1(X_1, X_2, \dots, X_\ell) + (X_2 - z_2) \cdot w_2(X_2, \dots, X_\ell) + s_2(X_3, X_4, \dots, X_\ell)$$

and so forth until we have written

$$q(X) - v = \sum_{i=1}^{\ell} (X_i - z_i) \cdot w_i(X_1, X_2, \dots, X_\ell) + s_\ell,$$

where s_ℓ depends on no variables, i.e., s_ℓ is simply an element in \mathbb{F}_p . Since $q(z) - v = 0$, it must hold that $s_\ell = 0$, completing the proof. \square

To open the commitment at input $z \in \mathbb{F}_p^\ell$ to value v , the prover computes w_1, \dots, w_ℓ as per Fact 13.2 and sends to the verifier values y_1, \dots, y_ℓ claimed to equal $g^{w_i(r)}$ for $i = 1, \dots, \ell$. Again, since each w_i is multilinear, $g^{w_i(r)}$ can be computed from the SRS despite the fact that the prover does not know r . The verifier checks that

$$e(c \cdot g^{-v}, g) = \prod_{i=1}^{\ell} e(y_i, g^{r_i} \cdot g^{-z_i}).$$

Note that the verifier is able to perform this check so long as the verification key includes g^{r_i} for each i (the verification key is a subset of the SRS, as each dictator function $(X_1, \dots, X_\ell) \mapsto X_i$ is a Lagrange basis polynomial).

Correctness is clear: if $c = g^{q(r)}$ and $y_i = g^{w_i(r)}$ for $i = 1, \dots, \ell$, then

$$e(c \cdot g^{-v}, g) = e(g^{q(r)-v}, g) = e(g^{\sum_{i=1}^{\ell} w_i(r) \cdot (r_i - z_i)}, g) = \prod_{i=1}^{\ell} e(g^{w_i(r)}, g^{r_i - z_i}) = e(y_i, g^{r_i} \cdot g^{-z_i}).$$

Here, the first inequality holds because $c = g^{q(r)}$, the second holds by Equation (13.23), the third holds by bilinearity of e , and the fourth holds because $y_i = g^{w_i(r)}$.

The proof of binding and techniques to achieve extractability are similar to the previous section and we omit them for brevity.

Costs. Like the univariate pairing-based polynomial commitment scheme of the previous section (Section 13.2.2), the ℓ -variate multilinear polynomial commitment consists of a constant number of group elements. However, whereas evaluation proofs for the univariate protocol also consisted of a constant number of group elements, evaluation proofs for the multilinear polynomial protocol are ℓ group elements rather than $O(1)$, with a corresponding increase in verification time from $O(1)$ group operations and bilinear map evaluations, to $O(\ell)$.

In terms of committer runtime, it turns out that the commitment scheme for multilinear polynomials has a particularly fast committer (faster than in the univariate protocol by a logarithmic factor in the number of coefficients of the polynomial being committed). Whereas the univariate protocol required fast polynomial division algorithms to compute the polynomial w needed during the evaluation phase (which require $\Theta(D \log D)$ field operations), Zhang et al. [ZGK⁺18, Appendix G] show that the committer in the protocol for multilinear polynomials can compute the polynomials w_1, \dots, w_ℓ with just $O(2^\ell)$ field operations in total. And once these polynomials are computed, the prover can compute all ℓ necessary values $g^{w_i(r)}$ with $O(2^\ell)$ many group exponentiations in total.

13.3 Commitment Scheme for Sparse Polynomials

Let us call a degree- D univariate polynomial *dense* if the number of nonzero coefficients is $\Omega(D)$, i.e., at least a constant fraction of the coefficients are nonzero. Similarly, call an ℓ -variate multilinear polynomial *dense* if the number of coefficients over the Lagrange basis is $\Omega(2^\ell)$. If a polynomial is not dense, we call it *sparse*.

An example of sparse polynomials that we have seen in this survey are $\widetilde{\text{add}}_i$ and $\widetilde{\text{mult}}_i$, the multilinear extensions of the functions add_i and mult_i that arose in our coverage of the GKR protocol (Section 4.6) and the related functions $\widetilde{\text{add}}$ and $\widetilde{\text{mult}}$ appearing in the MIP of Section 7.2. Indeed, $\widetilde{\text{add}}$ and $\widetilde{\text{mult}}$ are defined over $\ell = 3 \log |\mathcal{C}|$ variables, and the number of Lagrange basis polynomials over this many variables is $2^\ell = |\mathcal{C}|^3$. However, the number of nonzero coefficients of $\widetilde{\text{add}}$ and $\widetilde{\text{mult}}$ in the Lagrange basis is just $|\mathcal{C}|$.

As discussed in Section 4.6.6, when $\widetilde{\text{add}}$ and $\widetilde{\text{mult}}$ cannot be evaluated in time sublinear in $|\mathcal{C}|$, one technique to save the verifier time is for a trusted party to commit to these polynomials in a pre-processing phase with a polynomial commitment scheme. Then whenever the GKR protocol or MIP of Sections 4.6 and 7.2 (or SNARKs derived thereof) is applied to \mathcal{C} on a new input, the verifier need not evaluate $\widetilde{\text{add}}$ and $\widetilde{\text{mult}}$ on its own to perform the necessary checks of the prover's messages. Rather, the verifier can ask the prover to reveal the evaluations, using the evaluation phase of the polynomial commitment scheme. This reduces the verifier's runtime after the pre-processing phase from $\Theta(|\mathcal{C}|)$ to whatever is the verification time of the evaluation phase of the polynomial commitment scheme. Note that in this application of the polynomial commitment scheme, there is no need for the protocol to be zero-knowledge or extractable; it only needs to be binding to save the verifier work in the resulting zero-knowledge arguments for circuit satisfiability.

We have now seen several polynomial commitment schemes in which the committer’s runtime is dominated by doing a number of group exponentiations that is linear in the number of coefficients for *dense* univariate and multilinear polynomials (e.g., Section 13.1.3). However, these schemes do not offer any additional runtime savings for the committer if the polynomials are sparse. For example, applying these schemes directly to `add` and `mult` requires $\Omega(|\mathcal{C}|^3)$ time, which is totally impractical.¹⁴⁰

In this section, we describe a commitment scheme proposed by Setty [Set19] for any polynomial q such that the runtime of the committer is proportional to the *nonzero coefficients* M of q . The commitment scheme uses any polynomial commitment scheme for *dense, multilinear polynomials* as a subroutine.

For presentation purposes, we first describe a protocol that achieves the above goals up to a logarithmic factor. That is, the committer will have to apply a dense multilinear polynomial commitment scheme to a multilinear polynomial defined over $\ell' = \log_2 M + \log_2 \ell$ variables. This means that the committer has to perform $2^{\ell'} = O(M\ell)$ group exponentiations. Assuming M is $2^{\Omega(\ell)}$ (as is the case for `add` and `mult`), $O(M\ell) \leq O(M \log M)$. At the end of this section we sketch a technique to remove even this extra $\ell = \Theta(\log M)$ factor from the committer’s runtime by exploiting additional structure in `add` and `mult`.

The idea is to identify a layered arithmetic circuit \mathcal{C}' of fan-in two that takes as input a description of a sparse ℓ -variate multilinear polynomial q (we will specify the input description shortly) and a second input $z \in \mathbb{F}^\ell$, and such that \mathcal{C}' outputs $q(z)$. We will ensure that the input to \mathcal{C}' consists of $O(M\ell)$ field elements, and \mathcal{C}' has size $O(M\ell)$ and depth $O(\log M)$. Also, \mathcal{C}' will have wiring predicates `addi` and `multi` that can be evaluated any point in $O(\log(M\ell))$ time.

If s denotes the input to \mathcal{C}' specifying q , the commitment to q in our sparse polynomial commitment scheme will simply be a commitment to the multilinear extension \tilde{s} of s using any commitment scheme for dense multilinear polynomials. The reveal phase of our sparse polynomial commitment scheme works as follows. When the verifier requests the committer reveal $q(z)$ for a desired $z \in \mathbb{F}^\ell$, the committer sends the claimed value v of $q(z)$, and then the committer and verifier apply the GKR protocol to the claim that $\mathcal{C}'(s, z) = v$.¹⁴¹ At the end of the GKR protocol, the verifier needs to evaluate the multilinear extension of the input (s, z) at a random point. Since the verifier knows z but not s , using an observation analogous to Equation (6.1), the multilinear extension of (s, z) can be efficiently derived so long as the verifier learns an evaluation of \tilde{s} at a related point. The verifier can obtain this evaluation from the prover using the reveal phase of the dense polynomial commitment scheme.

Since \mathcal{C}' has size $O(M\ell)$, the GKR protocol prover applied to \mathcal{C}' can be implemented in $O(M\ell)$ total time, and committing to \tilde{s} using an appropriate dense polynomial commitment scheme requires $O(M\ell)$ group exponentiations. Since `addi` and `multi` for \mathcal{C}' can be evaluated in $O(\log(M\ell))$ time, the verifier’s runtime in the protocol is dominated by the cost of the evaluation phase of the dense polynomial commitment scheme.

Here is how the input to \mathcal{C}' will specify the polynomial q . Let $T_1, \dots, T_M \in \{0, 1\}^\ell$ denote the Lagrange basis polynomials $\chi_{T_1}, \dots, \chi_{T_M}$ that have nonzero coefficients c_1, \dots, c_M . That is, let

$$q(X) = \sum_{i=1}^M c_i \cdot \prod_{j=1}^{\ell} (T_{i,j} X_j + (1 - T_{i,j})(1 - X_j)). \quad (13.24)$$

The description s of q will consist of two lists $L[1], \dots, L[M]$ and $B[1], \dots, B[M]$, where $L[i] = c_i \in \mathbb{F}_p$ and $B[i] = T_i \in \{0, 1\}^\ell$. The circuit \mathcal{C}' will simply evaluate Equation (13.24) at input z . It is not hard to verify

¹⁴⁰Directly applying the pairing-based scheme for multilinear polynomials of Section 13.2.3 would allow the commitment to be computed in $O(|\mathcal{C}|)$ time, but the SRS would have length $|\mathcal{C}|^3$, and the evaluation phase may take time $\Omega(|\mathcal{C}|^3)$ for the committer.

¹⁴¹They could also apply the MIP of Section 7.2 to verify this claim, replacing the second prover with a polynomial commitment scheme for dense multilinear polynomials.

that Equation (13.24) can be evaluated by an arithmetic circuit with $O(M\ell)$ gates, such that the multilinear extensions of the wiring predicates add_i and mult_i for each layer of \mathcal{C}' can be evaluated in $O(\log(M\ell))$ time.

Saving a logarithmic factor by leveraging structure in $\widetilde{\text{add}}$ and $\widetilde{\text{mult}}$. The idea to shave a factor of ℓ from the size of \mathcal{C}' and the length of its input when committing to $q = \widetilde{\text{add}}$ or $q = \widetilde{\text{mult}}$ is as follows. First, we modify the description of q , so as to reduce the description length from $O(M \cdot \ell)$ field elements down to $O(M)$ field elements. Then we identify a Random Access Machine \mathcal{M} that takes as input this modified description of q and an input $z \in \mathbb{F}^\ell$ and outputs $q(z)$. We make sure that \mathcal{M} runs in time $O(M)$, and that \mathcal{M} can be transformed into a circuit of size that is just a constant factor larger than its runtime.

Here is how we modify the description of $q = \widetilde{\text{add}}$. Rather than having the identities of the nonzero Lagrange coefficients of add be specified via a list of bit-strings $T_1, \dots, T_M \in \{0, 1\}^{3 \log |\mathcal{C}|}$, we instead specify these identities with triples of integers $(u_1, u_2, u_3) \in \{1, \dots, |\mathcal{C}|\}^3$, and interpret a triple as indicating the u_1 'st gate of \mathcal{C} is an addition gate with in-neighbors assigned integer labels u_2 and u_3 .

Here is how we identify the Random Access Machine \mathcal{M} that, given the modified description of $q = \widetilde{\text{add}}$ and a point $z = (r_1, r_2, r_3) \in \left(\mathbb{F}_p^{\log |\mathcal{C}|}\right)^3$, runs in $O(M)$ time and outputs $q(z)$. Recall from Section 7.2 that $\text{add}(a, b, c): \{0, 1\}^{3 \log S} \rightarrow \{0, 1\}$ interprets its input as three gate labels a, b, c and outputs 1 if and only if b and c are the in-neighbors of gate a , and a is an addition gate. This means that

$$\widetilde{\text{add}}(X, Y, Z) = \sum_{a \in \{0, 1\}^{\log |\mathcal{C}|}} \chi_a(X) \cdot \chi_{\text{in}_1(a)}(Y) \cdot \chi_{\text{in}_2(a)}(Z), \quad (13.25)$$

where $\text{in}_1(a)$ and $\text{in}_2(a)$ respectively denote the labels in $\{0, 1\}^{\log |\mathcal{C}|}$ of the first and second in-neighbors of gate a . Recall in addition from Lemma 3.8 that evaluating all $(\log |\mathcal{C}|)$ -variate Lagrange basis polynomials at a specified input $r \in \mathbb{F}^{\log |\mathcal{C}|}$ can be done in $O(|\mathcal{C}|)$ time. So to evaluate $\widetilde{\text{add}}$ at an input $(r_1, r_2, r_3) \in \left(\mathbb{F}^{\log |\mathcal{C}|}\right)^3$ in $O(|\mathcal{C}|)$ time, it suffices for \mathcal{M} to operate in two phases. In the first phase, \mathcal{M} evaluates all $(\log |\mathcal{C}|)$ -variate Lagrange basis polynomials at the three inputs r_1, r_2, r_3 in $O(|\mathcal{C}|)$ time (this can be done without even examining the list of triples (u_1, u_2, u_3)), and stores the $3 \cdot |\mathcal{C}|$ results in memory. In the second phase, \mathcal{M} evaluates add at (r_1, r_2, r_3) via Equation (13.25) in $O(|\mathcal{C}|)$ additional time, given random access to the memory contents.

Using the computer-programs-to-circuit-satisfiability transformation of Chapter 5, specifically using the fingerprinting-based memory-checking procedure described in Section 5.6.2, \mathcal{M} can be transformed into a circuit-satisfiability instance for a circuit \mathcal{C}' (as described in Sections 5.6.2 and 5.6.3, the transformation procedure from \mathcal{M} to the circuit-satisfiability instance is itself interactive, but the transformation can be rendered non-interactive in the random oracle model using the Fiat-Shamir transformation).

13.4 Polynomial Commitment Schemes: Pros and Cons

We have seen three approaches to the construction of practical polynomial commitment schemes. The first is based on IOPs, such as the IOP for Reed-Solomon testing called FRI (Section 9.4). The second (Section 13.1) builds in highly sophisticated ways on homomorphic commitment schemes such as Pedersen commitments [Ped91], and Schnorr-style [Sch89] techniques for proving inner product relations between a committed vector and a public vector; this approach based binding on the assumed hardness of the discrete logarithm problem. The third (Section 13.2.2) is derived from work of KZG [KZG10] and is based on bilinear maps and requires a trusted setup. Roughly, the practical pros and cons of the three approaches to polynomial commitment schemes are the following.

Pros and cons of the IOP-based polynomial commitments. The IOP approach is the only one of the three approaches that is plausibly quantum-secure (it is secure in the quantum random oracle model [CMS19]). The other two approaches both assume hardness of the discrete logarithm problem, which is a problem that quantum computers can solve in polynomial time. Another advantage of the IOP approach is that it uses very small public parameters (these simply specify one or more cryptographic hash functions) that moreover can be generated at random (i.e., they simply specify a randomly chosen hash function from a collision-resistant hash family). That is, unlike the third approach, the first (and also the second) approach is transparent, meaning it does not require a structured reference string (SRS) generated by a trusted party who will have the power to forge proofs of evaluations if they fail to discard toxic waste.

The downsides of the IOP approach include the following. First, in order to commit to a polynomial with n coefficients, this approach requires the prover to perform a Fast Fourier Transform over a vector of length $O(n)$ in order to evaluate the polynomial being committed at $\Omega(n)$ many points. Such FFTs are time-intensive ($O(n \log n)$ field operations), as well as space intensive and difficult to parallelize and distribute. The need to perform FFTs also forces the polynomial to be defined over a finite field that supports efficient FFT algorithms, though this is not a major limitation as many fields support quasilinear time FFT algorithms. Second, the reveal phase of this polynomial commitment scheme requires $O(\log n)$ many Merkle-tree authentication paths to be sent to the verifier; each such path consists of $O(\log n)$ hash values, and this can result in concretely long proofs (while each hash value is typically only a few hundred bits, many hash values are sent to the verifier). Known soundness analyses of this approach currently require many repetitions in the evaluation phase, further increasing proof length (see Section 9.4). Third, the other two approaches yield homomorphic commitments (i.e., given two commitments c_p, c_q to two polynomials p and q , a commitment c to the sum of $p + q$ can be derived). This homomorphism property is essential in some applications of polynomial commitment schemes (see for example Section 15.5 later in this survey).

Pros and cons of discrete-logarithm based polynomial commitments. The second approach (Section 13.1.3) also does not require trusted setup because the public parameters are simply random group elements. Since this approach does not require FFTs, and the committer performs $O(n)$ many group exponentiations in any group for which the discrete logarithm problem is hard, this approach currently has the best known concrete efficiency for the committer. Until the advent of Dory (Section 13.1.5), this approach did require larger verifier runtime than the other two approaches. For example, in the implementation of this approach described in Section 13.1.3, the verifier does $O(\sqrt{n})$ group exponentiations. Dory reduces the verifier time to $O(\log n)$ group exponentiations, at the cost of constant-factor increase in prover time (much of this constant factor is due to Dory’s need to operate in a pairing-friendly group, which can lead concretely slower group operations).

Pros and cons of KZG-based polynomial commitments. The primary benefit of the third approach is that, when applied to univariate polynomials (Section 13.2.2), both the commitment and proofs of evaluation consist of a constant number of group elements, and can be verified with a constant number of group operations and one bilinear map evaluation (for multilinear polynomials (Section 13.2.3) these costs are logarithmic in the number of coefficients instead of constant).

A significant downside of the third approach is that it requires trusted setup (an SRS, with toxic waste that must be discarded to avoid forgeability of evaluation proofs). There has been significant work to mitigate the trust assumptions required, and it is now known how to make the SRS “updatable”. This means that the SRS can be updated at any point by any party, such that if even a single party is honest (meaning a single party discards their toxic waste, then no one can forge proofs of evaluation) [MBKM19]. The rough idea for

why this is possible is that the SRS consists of powers of a random nonzero field element $\tau \in \{1, \dots, p-1\}$ in the exponent of a group generator g , so any party can “rerandomize the choice of τ ” by picking a random $s \in \{1, \dots, p-1\}$ and updating the i th element of the SRS from g^{τ^i} to $(g^{\tau^i})^{s^i} = g^{(\tau s)^i}$. That is, by raising the i th entry of the SRS to the power s^i , τ is effectively updated to $\tau \cdot s \pmod p$, which is a random element of $\{1, \dots, p-1\}$.

The third approach is also computation-intensive, for multiple reasons. For example, it requires a similar number of public-key cryptographic operations (i.e., group exponentiations) as the second approach, but as with Dory these operations must be in pairing-friendly groups, for which group operations are computationally more intensive. Second, in the univariate setting it requires the prover to perform polynomial division, which is an operation with an efficiency profile similar to the FFTs utilized in Approach 1 (for multilinear polynomials, polynomial division is not required; see the discussion of Costs in Section 13.2.3).

13.5 Additional Approaches

Recent polynomial commitment schemes that we do not discuss in this survey include [BFS20], which is based on a cryptographic notion called groups of unknown order. This work (which predates the Dory polynomial commitment scheme covered in Section 13.1.5) achieves similar verifier complexity to Dory (logarithmic time), but the prover is significantly slower than Dory both asymptotically and concretely. Another example is recent work [BBC⁺18, BLNS20] that operates in a manner similar to Bulletproofs (Section 13.1.4), but modifies it in order to base security on lattice assumptions that are believed to be post-quantum secure. This approach currently appears to yield significantly larger proofs than the discrete-logarithm based protocols that it is inspired by.

Chapter 14

Linear PCPs and Succinct Arguments

14.1 Overview: Interactive Arguments From “Long”, Structured PCPs

As discussed in Section 8.4, short PCPs for circuit satisfiability tend to be quite complicated (recall that, by short, we mean of length quasilinear in the circuit size). Ishai, Kushilevitz, and Ostrovsky [IKO07] (IKO) put forth an ingenious two-step methodology for developing argument systems for arithmetic circuit satisfiability, without the use of short PCPs. The basic idea is the following. Why is the prover inefficient if one instantiates Kilian’s argument system [Kil92] with a PCP of superpolynomial length $L = n^{\omega(1)}$? The problem is that \mathcal{P} has to materialize the full proof π in order to compute its Merkle-hash and thereby “commit” to π . Materializing a proof of superpolynomial length clearly takes superpolynomial time.

But IKO [IKO07] show that if π is highly structured in a manner made precise below, then there is a way for \mathcal{P} to cryptographically commit to π *without* materializing all of it. This enables IKO to use structured PCPs of exponential length to obtain succinct interactive arguments. Such “long” PCPs turn out to be much simpler to construct than short PCPs. The commitment protocol of [IKO07] is based on any semantically secure, additively-homomorphic cryptosystem, such as ElGamal encryption [ElG85]. Here, an additively-homomorphic encryption scheme is analogous to the notion of an additively-homomorphic commitment scheme (Section 11.3) which we exploited at length in the zero-knowledge arguments of Chapter 12. It is an encryption scheme for which one can compute addition over encrypted data without decrypting the data. Later work [GGPR13, BCI⁺13] gave a different technique, based on pairings, for transforming such long, structured PCPs into *non-interactive*, publicly-verifiable arguments (zk-SNARKs).

A downside of long PCPs is that if the proof π has length L , then even writing down one of the verifier’s queries to π requires $\log L$ field elements. If L is exponential in the circuit size $S = |\mathcal{C}|$, then even $\log(L)$ is *linear* in the circuit size. This means that even writing down the verifier’s message takes time $\Omega(S)$. Compare this to the MIPs, PCPs, and IOPs from previous sections, where the verifier’s total runtime was $O(n + \text{polylog}(S))$. In the zk-SNARKs we obtain in this section, these long messages from the PCP verifier will translate into a long structured reference string (of length $\Omega(S)$) that must be generated in a trusted manner.

However, these downsides are mitigated by the ability of the verifier to amortize this large cost when outsourcing the evaluation of \mathcal{C} on many separate inputs. In addition, the communication in the reverse direction of the linear PCP, from prover to verifier, is very small (a constant number of field elements) and the verifier’s online verification phase is especially fast as a consequence (\mathcal{V} performs just $O(1)$ field or group operations to check \mathcal{P} ’s final message). Because of these advantages, zk-SNARKs derived from linear PCPs have seen deployment in blockchain and cryptocurrency applications. Proof size is of paramount importance

in these applications because proofs must be stored on the blockchain indefinitely.

Linear PCPs. The type of structure that IKO [IKO07] exploit is linearity. Specifically, in a *linear PCP*, the proof is interpreted as a function mapping $\mathbb{F}^v \rightarrow \mathbb{F}$ for some integer $v > 0$. A linear PCP is one in which the “honest” proof is a linear function π . That is, π should satisfy that for any two queries $q_1, q_2 \in \mathbb{F}^v$ and constants $d_1, d_2 \in \mathbb{F}$, $\pi(d_1q_1 + d_2q_2) = d_1\pi(q_1) + d_2\pi(q_2)$. This is the same as requiring that π be a v -variate polynomial of total degree 1, with constant term equal to 0. Note that in a linear PCP, soundness should hold even against “cheating” proofs that are non-linear. So the only difference between a linear PCP and a PCP is that in a linear PCP, the honest proof is guaranteed to have special structure.

The arguments of IKO [IKO07] conceptually proceed in the same two steps as Kilian’s argument based on short PCPs (Section 8.2). In the first step, the prover commits to the proof π , but unlike in Kilian’s approach, here the prover can leverage the linearity of π to commit to it without ever materializing π in full. In the second step, the argument system verifier simulates the linear PCP verifier, asking the prover to reveal certain locations of the proof, which the prover does using the reveal phase of the commitment protocol.

Hence, in order to give an efficient argument system, IKO [IKO07] had to do two things. First, they gave a commit/reveal protocol for linear functions. Second, they gave a linear PCP for arithmetic circuit satisfiability.

Unfortunately, when applied to circuits of size S , their linear PCP had length $|\mathbb{F}|^{O(S^2)}$. It takes $O(S^2)$ time for the verifier to even *write down* a query into a proof of this length. A subsequent refinement of Gennaro, Gentry, Parno, and Raykova (GGPR) [GGPR13] give a linear PCP¹⁴² of length $|\mathbb{F}|^{O(S)}$. GGPR’s linear PCP has formed the theoretical foundation for many of the argument systems that have been implemented in the research literature and deployed in commercial settings.

Section 14.2 covers IKO’s commit/reveal protocol for linear functions. Section 14.3 describes IKO’s linear PCP of length $|\mathbb{F}|^{O(S^2)}$. Section 14.4 covers GGPR’s linear PCP of length $|\mathbb{F}|^{O(S)}$. Section 14.5 explains how to translate GGPR’s linear PCP into a SNARK using pairing-based cryptography.

14.2 Committing to a Linear PCP without Materializing It

Let π be a linear function $\mathbb{F}^v \rightarrow \mathbb{F}$. This section sketches the technique of IKO [IKO07] for allowing the prover to first commit to π in a “commit phase” and then answer a series of k queries $q^{(1)}, \dots, q^{(k)} \in \mathbb{F}^v$ in a “reveal phase”. Roughly speaking, the security guarantee is that at the end of the commit phase, there is *some* function π' (which may not be linear) such that, if the verifier’s checks in the protocol all pass and \mathcal{P} cannot break the cryptosystem used in the protocol, then the prover’s answers in the reveal phase are all consistent with π' .¹⁴³

In more detail, the protocol uses a semantically secure homomorphic cryptosystem. Roughly speaking, semantic security guarantees that, given a ciphertext $c = \text{Enc}(m)$ of a plaintext m , any probabilistic polynomial time algorithm cannot “learn anything” about m . Semantic security is an analog of the hiding property of commitment schemes such as Pedersen commitments (Section 11.3). A cryptosystem is (additively) homomorphic if, for any pair of plain texts (m_1, m_2) and fixed constants $d_1, d_2 \in \mathbb{F}$, it is possible to efficiently compute the encryption of $d_1m_1 + d_2m_2$ from the encryptions of m_1 and m_2 individually. Here, in the context

¹⁴²The observation that GGPR’s protocol is actually a linear PCP as defined by IKO was made in later work [BCI⁺13, SBV⁺13].

¹⁴³This section actually sketches a refinement of IKO’s commitment/reveal protocol, due to Setty et al. [SMBW12]. The original protocol of IKO guaranteed that for each query i , there is a separate function π_i to which \mathcal{P} was committed. Setty et al. [SMBW12] tweaked the protocol of IKO in a way that both reduced costs and guaranteed that \mathcal{P} was committed to answering all k queries using a single function π' (which is possibly non-linear).

of linear PCPs, m_1 and m_2 will be elements of the field \mathbb{F} , so the expression $d_1m_1 + d_2m_2$ refers to addition and scalar multiplication over \mathbb{F} .

Many additively homomorphic encryption schemes are known, such as the popular ElGamal encryption scheme, whose security is based on the Decisional Diffie-Hellman assumption introduced in Section 13.2.1.

In the commit phase, the verifier chooses a vector $r = (r_1, \dots, r_v) \in \mathbb{F}^v$ at random, encrypts each entry of r and sends all v encryptions to the prover. Since π is linear, there is some vector $d = (d_1, \dots, d_v) \in \mathbb{F}^v$ such that $\pi(q) = \sum_{i=1}^v d_i \cdot q_i = \langle d, q \rangle$ for all queries $q = (q_1, \dots, q_v)$. Hence, using the homomorphism property of the encryption scheme, the prover can efficiently compute the encryption of $\pi(r)$ from the encryptions of the individual entries of r . Specifically, $\text{Enc}(\pi(r)) = \text{Enc}(\sum_{i=1}^v d_i r_i)$, and by the homomorphism property of Enc , this last expression can be efficiently computed from $\text{Enc}(r_1), \dots, \text{Enc}(r_v)$. The prover sends this encryption to the verifier, who decrypts it to obtain (what is claimed to be) $s = \pi(r)$.

Remark 14.1. At this point in the protocol, using the homomorphism property of Enc and the linearity of π , the honest prover has managed to send to the verifier an encryption of $\pi(r)$, even though the prover has no idea what r is (this is what semantic security of Enc guarantees). Moreover, the prover has accomplished this in $O(v)$ time. This is far less than the $\Omega(|\mathbb{F}|^v)$ time required to evaluate π at all points, which would be required if the prover were to build a Merkle tree with the evaluations of π as the leaves.

One may wonder whether the use of an additively homomorphic *encryption* scheme Enc can be replaced with an additively homomorphic *commitment* scheme such as Pedersen commitments. Indeed, given a Pedersen commitment to each entry of r , the prover could compute a Pedersen commitment c^* to $\pi(r)$ using additive homomorphism, despite not knowing r , just as the prover in this section is able to compute $\text{Enc}(\pi(r))$ given $\text{Enc}(r_1), \dots, \text{Enc}(r_v)$. The problem with this approach is that the verifier, who does not know π , would not be able to open c^* to $\pi(r)$. In contrast, by using an encryption scheme, the verifier can decrypt $\text{Enc}(\pi(r))$ to $\pi(r)$ despite not knowing π .

In the reveal phase, the verifier picks k field elements $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ at random, and keeps them secret. The verifier then sends the prover the queries $q^{(1)}, \dots, q^{(k)}$ in the clear, as well as $q^* = r + \sum_{i=1}^k \alpha_i \cdot q^{(i)}$. The prover returns claimed answers $a^{(1)}, \dots, a^{(k)}, a^* \in \mathbb{F}$, which are supposed to equal $\pi(q^{(1)}), \dots, \pi(q^{(k)}), \pi(q^*)$. The verifier checks that $a^* = s + \sum_{i=1}^k \alpha_i \cdot a^{(i)}$, accepting the answers as valid if so, and rejecting otherwise.

Clearly the verifier's check will pass if the prover is honest. The proof of binding roughly argues that the only way for \mathcal{P} to pass the verifier's checks, if \mathcal{P} does not answer all queries using a single function, is to know the α_i 's, in the sense that one can efficiently compute the α_i 's given access to such a prover. But if the prover knows the α_i 's, then the prover must be able to solve for r , since \mathcal{V} reveals $q^* = r + \sum_{i=1}^k \alpha_i \cdot q^{(i)}$ to the prover. But this would contradict the semantic security of the underlying cryptosystem, which guarantees that the prover learned nothing about r from the encryptions of r 's entries.

14.2.1 Detailed Presentation of Binding Property When $k = 1$

We present the main idea of the proof of binding, in the case the $k = 1$.

What does it mean for the prover *not* to be bound to a fixed function after the commitment phase of the protocol? It means that there are at least two runs of the reveal protocol, where in the first run, the verifier sends queries q_1 and $q^* = r + \alpha \cdot q_1$, and the prover responds with answers a_1 and a^* , while in the second run the verifier sends queries q_1 and $\hat{q} = r + \alpha' \cdot q_1$, and the prover responds with answers $a'_1 \neq a_1$ and \hat{a} . That is, in two different runs of the reveal protocol, the prover responded to the same query q_1 with two different answers, and managed to pass the verifier's checks.

As indicated above, we will argue that in this case, the prover must know α and α' . But, as we now explain, this breaks the semantic security of the encryption scheme.

Why the prover knowing α and α' means semantic security is broken. Roughly speaking, this is because if the prover really learned nothing about r from $\text{Enc}(r_1), \dots, \text{Enc}(r_v)$, as promised by semantic security of Enc , then it should be impossible for the prover to determine α with probability noticeably better than random guessing, even given $q_1, q^* = r + \alpha \cdot q_1$, and $\hat{q} = r + \alpha' \cdot q_1$. This is because, without knowing r , all that the equations

$$q^* = r + \alpha q_1 \tag{14.1}$$

and

$$\hat{q} = r + \alpha' \cdot q_1 \tag{14.2}$$

tell the prover about α and α' is that they are two field elements satisfying $q^* - \hat{q} = (\alpha - \alpha')q_1$. That is, for every pair $\alpha, \alpha' \in \mathbb{F}$ satisfying Equations (14.1) and (14.2) for r , and any $c \in \mathbb{F}$, the pair $\alpha + c, \alpha' + c$ also satisfy both equations when r is replaced by $r + cq_1$. So without knowing anything about r , Equations (14.1) and (14.2) reveal no information whatsoever about α itself. Equivalently stated, if the prover knows α , then the prover must have learned something about r , in violation of semantic security of Enc .

Showing that the prover must know α and α' . Recall that s is the decryption of the value sent by the prover in the commit phase, which is claimed to be $\text{Enc}(\pi(r))$. Since the prover cannot decrypt, the prover does not know s . Even so, if the verifier's checks in the two runs of the reveal phase pass, then the prover does know that:

$$a^* = s + \alpha a_1, \tag{14.3}$$

and

$$\hat{a} = s + \alpha' a_1. \tag{14.4}$$

Subtracting these two equations means that the prover knows that

$$(a^* - \hat{a}) = \alpha a_1 - \alpha' a_1, \tag{14.5}$$

Similarly, even though the prover doesn't know r , the prover does know that Equations (14.1) and (14.2) hold. Subtracting those two equations implies that $q^* - \hat{q} = (\alpha - \alpha')q_1$.

We may assume that none of the queries are the all-zeros vector, since any linear function π evaluates to 0 on the all-zeros vector. Hence, if we let j denote any nonzero coordinate of q_1 , then:

$$q_j^* - \hat{q}_j = (\alpha - \alpha')q_{1,j}. \tag{14.6}$$

Since $a_1 \neq a'_1$, Equations (14.5) and (14.6) express α and α' via two linearly independent equations in two unknowns, and these have a unique solution. Hence, the prover can solve these two equations for α and α' as claimed.

14.3 A First Linear PCP for Arithmetic Circuit Satisfiability

Let $\{\mathcal{C}, x, y\}$ be an instance of arithmetic circuit satisfiability (see Section 5.5.1). For this section, we refer to a setting $W \in \mathbb{F}^S$ of values to each gate in \mathcal{C} as a transcript for \mathcal{C} .

The linear PCP of this section is from IKO [IKO07], and is based on the observation that W is a correct transcript if and only if W satisfies the following $\ell = S + |y| - |w|$ constraints (there is one constraint for every other non-output gate of \mathcal{C} , there are two constraints for each output gate of \mathcal{C} , and there are no constraints for any witness elements).

- For each input gate a , there is a constraint enforcing that $W_a - x_a = 0$. This effectively insists that the transcript W actually corresponds to the execution of \mathcal{C} on input x , and not some other input.
- For each output gate a there is a constraint enforcing that $W_a - y_a = 0$. This effectively insists that the transcript W actually correspond to an execution of \mathcal{C} that produces outputs y , and not some other set of outputs.
- If gate a is an addition gate with in-neighbors $\text{in}_1(a)$ and $\text{in}_2(a)$, there is a constraint enforcing that $W_a - (W_{\text{in}_1(a)} + W_{\text{in}_2(a)}) = 0$.
- If gate a is a multiplication, there is a constraint enforcing that $W_a - W_{\text{in}_1(a)} \cdot W_{\text{in}_2(a)} = 0$.

Together, the last two types of constraints insist that the transcript actually respects the operations performed by the gates of \mathcal{C} . That is, any addition (respectively, multiplication) gate actually computes the addition (respectively, product) of its two inputs. Note that the constraint for gate a of \mathcal{C} is always of the form $Q_a(W) = 0$ for some polynomial Q_a of degree at most 2 in the entries of W .

For a transcript W for $\{\mathcal{C}, x, y\}$, let $W \otimes W$ denote the length- (S^2) vector whose (i, j) th entry is $W_i \cdot W_j$. Let $(W, W \otimes W)$ denote the vector of length $S + S^2$ obtained by concatenating W with $W \otimes W$. Let

$$f_{(W, W \otimes W)}(\cdot) := \langle \cdot, (W, W \otimes W) \rangle.$$

That is, $f_{(W, W \otimes W)}$ is the linear function that takes as input a vector in \mathbb{F}^{S+S^2} and outputs its inner product with $(W, W \otimes W)$. Consider a linear PCP proof π containing all evaluations of $f_{(W, W \otimes W)}$. π is typically called the *Hadamard encoding* of $(W, W \otimes W)$. Notice that π has length $|\mathbb{F}|^{S+S^2}$, which is enormous. However, \mathcal{P} will never need to explicitly materialize all of π .

\mathcal{V} needs to check three things. First, that π is a linear function. Second, assuming that π is a linear function, \mathcal{V} needs to check that π is of the form $f_{(W, W \otimes W)}$ for some transcript W . Third, \mathcal{V} must check that W satisfies all ℓ constraints described above.

First Check: Linearity Testing. Linearity testing is a considerably simpler task than the more general low-degree testing problems considered in the MIP of Section 7.2 (linearity testing is equivalent to testing that an m -variate function equals polynomial of total degree 1 (with no constant term), while the low-degree testing problem considered in Section 7.2 tested whether an m -variate function is multilinear, which means its total degree can be as large as m).

Specifically, to perform linearity testing, the verifier picks two random points $q^{(1)}, q^{(2)} \in \mathbb{F}^{S+S^2}$ and checks that $\pi(q^{(1)} + q^{(2)}) = \pi(q^{(1)}) + \pi(q^{(2)})$, which requires three queries to π . If π is linear then the test always passes. Moreover, it is known that if the test passes with probability $1 - \delta$, then there is some linear function f_d such that π is δ -close to f_d [BLR93], at least over fields of characteristic 2.¹⁴⁴

¹⁴⁴See [AB09, Theorem 19.9] for a short proof of this statement based on Discrete Fourier analysis. Over fields of characteristic other than 2, the known soundness guarantees of the linearity test are weaker. See [SBV⁺13, Proof of Lemma A.2] and [BCH⁺96, Theorem 1.1].

Second Check. Assuming π is linear, π can be written as f_d for some vector $d \in \mathbb{F}^{S+S^2}$. To check that d is of the form $(W, W \otimes W)$ for some transcript W , \mathcal{V} does the following.

- \mathcal{V} picks $q^{(3)}, q^{(4)} \in \mathbb{F}^S$ at random.
- Let $(q^{(3)}, \mathbf{0})$ denote the vector in \mathbb{F}^{S+S^2} whose first S entries equal $q^{(3)}$ and whose last S^2 entries are 0. Similarly, let $(\mathbf{0}, q^{(3)} \otimes q^{(4)})$ denote the vector whose first S entries equal 0, and whose last S^2 entries equal $q^{(3)} \otimes q^{(4)}$. \mathcal{V} checks that $\pi(q^{(3)}, \mathbf{0}) \cdot \pi(q^{(4)}, \mathbf{0}) = \pi(\mathbf{0}, q^{(3)} \otimes q^{(4)})$. This requires three queries to π .

Clearly the check will pass if π is of the claimed form. If π is not of the claimed form, the test will fail with high probability over the choice of $q^{(3)}$ and $q^{(4)}$. This holds because $\pi(q^{(3)}, \mathbf{0}) \cdot \pi(q^{(4)}, \mathbf{0}) = f_d(q^{(3)}, \mathbf{0}) \cdot f_d(q^{(4)}, \mathbf{0})$ is a quadratic polynomial in the entries of $q^{(3)}$ and $q^{(4)}$, as is $f_d(\mathbf{0}, q^{(3)} \otimes q^{(4)})$, and the Schwartz-Zippel lemma (Lemma 3.3) guarantees that any two distinct low-degree polynomials can agree on only a small fraction of points.

Third Check. Once \mathcal{V} is convinced that $\pi = f_d$ for some d of the form $(W, W \otimes W)$, \mathcal{V} is ready to check that W satisfies all ℓ constraints described above. This is the core of the linear PCP.

In order to check that $Q_i(W) = 0$ for all constraints i , it suffices for \mathcal{V} to pick random values $\alpha_1, \dots, \alpha_\ell \in \mathbb{F}$, and check that $\sum_{i=1}^{\ell} \alpha_i Q_i(W) = 0$. Indeed, this equality is always satisfied if $Q_i(W) = 0$ for all i ; otherwise, $\sum_{i=1}^{\ell} \alpha_i Q_i(W)$ is a nonzero multilinear polynomial in the variables $(\alpha_1, \dots, \alpha_\ell)$, and the Schwartz-Zippel lemma guarantees that this polynomial is nonzero at almost all points $(\alpha_1, \dots, \alpha_\ell) \in \mathbb{F}^\ell$.

Notice that $\sum_{i=1}^{\ell} \alpha_i Q_i(W)$ is itself a degree-2 polynomial in the entries of W , which is to say that it is a linear combination of the entries of $(W, W \otimes W)$. Hence it can be evaluated with one additional query to π .

Soundness Analysis. A formal proof of the soundness of the linear PCP just described is a bit more involved than indicated above, but not terribly so. Roughly it proceeds as follows. If the prover passes the linearity test with probability $1 - \delta$, then π is δ -close to a linear function f_d . Hence, as long as the 4 queries in the second and third checks are distributed uniformly in \mathbb{F}^{S+S^2} , then with probability at least $1 - 4 \cdot \delta$, the verifier will never encounter a point where π and f_d differ, and we can treat π as f_d for the remainder of the analysis. However, the queries in the second and third checks are not uniformly distributed in \mathbb{F}^{S+S^2} as described. Nonetheless, they can be made uniformly distributed by replacing each query q with two random queries q' and q'' subject to the constraint that $q' + q'' = q$. This way, the marginal distributions of q' and q'' are uniform over \mathbb{F}^{S+S^2} . And by linearity of f_d , it holds that $f_d(q)$ can be deduced to equal $f_d(q') + f_d(q'')$. With this change, the soundness analysis of the second and third steps are as indicated above.

Protocol Costs. The costs of the argument system obtained by combining the above linear PCP with the commitment protocol are summarized in Table 14.1. \mathcal{V} 's time and \mathcal{P} 's time are both $\Theta(S^2)$, but if \mathcal{V} is simultaneously verifying \mathcal{C} 's execution over a large *batch* of inputs, then the $\Theta(S^2)$ cost for \mathcal{V} can be amortized over the entire batch. Total communication from \mathcal{V} to \mathcal{P} is $\Theta(S^2)$ as well (this cost can also be amortized). Such $\Theta(S^2)$ costs are very high, precluding practicality. On the positive side, the communication in the reverse direction is just a constant number of field elements per input. Section 14.4 below explains how to drastically reduce \mathcal{V} 's and \mathcal{P} 's runtimes, and the communication from \mathcal{V} to \mathcal{P} , from $\Theta(S^2)$ to $\Theta(S)$.

$\mathcal{V} \rightarrow \mathcal{P}$ Communication	$\mathcal{P} \rightarrow \mathcal{V}$ Communication	Queries	\mathcal{V} time	\mathcal{P} time
$O(S^2)$ field elements	$O(1)$ field elements	$O(1)$	$O(S^2)$	$O(S^2)$

Table 14.1: Costs of the argument system from Section 14.3 for arithmetic circuit satisfiability when run on a circuit \mathcal{C} of size S . Note that the verifier’s cost and the communication cost can be amortized when simultaneously outsourcing \mathcal{C} ’s execution on a large *batch* of inputs. The stated bound on \mathcal{P} ’s time assumes \mathcal{P} knows a witness w for \mathcal{C} .

14.4 GGPR: A Linear PCP Of Size $O(|\mathbb{F}|^S)$ for Arithmetic Circuit-SAT

In a breakthrough result, Gennaro, Gentry, Parno, and Raykova [GGPR13] gave a linear PCP of length $O(|\mathbb{F}|^S)$ for arithmetic circuit satisfiability, where S denotes the size of the circuit. In fact, their linear PCP also solves the more general problem called RICS satisfiability (see Section 7.5 for discussion of RICS satisfiability). Gennaro et al. referred to RICS satisfiability problems as *Quadratic Arithmetic Programs* (QAPs).^{145,146} We restrict our presentation here to arithmetic circuits for simplicity. The linear PCP of [GGPR13] have been highly influential, and form the foundation of many of the implementations of argument systems.

QAPs use the same constraint-based formalism as the linear PCP described in Section 14.3. Recall that there are $\ell = S + |y| - |w|$ constraints $Q_i(W) = 0$, where Q_i is a polynomial in the variables of W that always takes one of three forms. The three forms are: (1) $W_i - c_i = 0$ for some value c_i depending on the public input x or outputs y , (2) $W_i - (W_j \cdot W_k) = 0$, or (3) $W_i - (W_j + W_k) = 0$. A crucial observation is that in all three cases, Q_i can always be written in the form $f_{1,i}(W) \cdot f_{2,i}(W) - f_{3,i}(W) = 0$, for some affine¹⁴⁷ functions $f_{1,i}$, $f_{2,i}$, and $f_{3,i}$. This is a stronger notion of structure than was exploited Section 14.3 (Section 14.3 only exploited that each constraint is a polynomial in W of total degree at most 2).

Let $H := \{\sigma_1, \dots, \sigma_\ell\}$ be a set of ℓ arbitrary distinct elements in \mathbb{F} . Our ultimate goal is to associate any transcript W with a univariate polynomial $g_{x,y,W}(t)$ that vanishes on H if and only if W is a correct transcript. To define $g_{x,y,W}$, we must first define several constituent polynomials that together capture the above constraints.

Transforming constraints into polynomials. For each gate i in \mathcal{C} , define three univariate polynomials A_i , B_i , and C_i , each of degree $\ell - 1$, via interpolation as follows.

$$A_i(\sigma_j) = \text{the coefficient of } W_i \text{ in } f_{1,j}.$$

$$B_i(\sigma_j) = \text{the coefficient of } W_i \text{ in } f_{2,j}.$$

$$C_i(\sigma_j) = \text{the coefficient of } W_i \text{ in } f_{3,j}.$$

Finally, define via interpolation 3 univariate polynomials of degree $\ell - 1$ via interpolation as follows.

$$A'(\sigma_j) = \text{the constant term in } f_{1,j}.$$

$$B'(\sigma_j) = \text{the constant term in } f_{2,j}.$$

$$C'(\sigma_j) = \text{the constant term in } f_{3,j}.$$

¹⁴⁵The argument system of Gennaro et al. can be understood in multiple ways, and [GGPR13] did not present it within the framework of linear PCPs. Subsequent work [SBV⁺13, BCI⁺13] identified the protocol of Gennaro et al. as an example of a linear PCP.

¹⁴⁶The focus of Gennaro et al. [GGPR13] was on the development of *non-interactive* argument systems satisfying various additional properties, especially zero-knowledge. We will describe such non-interactive arguments in Section 14.5. The nonzero-knowledge interactive argument from this section was described and implemented by Setty et al. [SBV⁺13].

¹⁴⁷Affine means that the functions are of the form $\pi + c$ for some linear function π and constant c .

Example. Suppose that x and w each consist of one field element and $\mathcal{C}(x, w)$ consists of a single multiplication gate, i.e., $\mathcal{C}(x, w) = x \cdot w$. Suppose that $x = 4$ and the prover claims to know a w such that $\mathcal{C}(x, w) = 9$ (clearly, the verifier can easily compute w unaided as $w = 9 \cdot 4^{-1}$, but let us ignore this for purposes of this example, which is merely meant to illustrate the definition of the polynomials above). A transcript W for \mathcal{C} then consists of three variables, $W = (W_1, W_2, W_3)$, where W_1 corresponds to the value of the public input x , W_2 to the witness element w , and W_3 to the value of the multiplication gate when evaluated on input (x, w) .

Then there are three constraints that together capture whether or not W is a correct transcript for the claim that there exists a w such that $\mathcal{C}(x, w) = 9$: (1) $W_1 - 4 = 0$, (2) $W_3 - W_1 \cdot W_2 = 0$, (3) $W_3 - 9 = 0$. These three constraints define the following nine affine functions, three per constraint (there is flexibility in how these affine functions are chosen):

- $f_{1,1}(W) = 0, f_{2,1}(W) = 0, f_{3,1}(W) = -W_1 + 4,$
- $f_{1,2}(W) = W_1, f_{2,2}(W) = W_2, f_{3,2}(W) = W_3,$
- $f_{1,3}(W) = 0, f_{2,3}(W) = 0, f_{3,3}(W) = -W_3 + 9.$

These affine functions then uniquely define the following twelve polynomials (three per variable of W , in addition to A', B', C'):

- $A_1(\sigma_2) = 1$ and $A_1(\sigma_1) = A_1(\sigma_3) = 0$. That is,

$$A_1(X) = (\sigma_2 - \sigma_1)^{-1}(\sigma_2 - \sigma_3)^{-1} \cdot (X - \sigma_1)(X - \sigma_3).$$

- $C_1(\sigma_1) = -1$ while $C_1(\sigma_2) = C_1(\sigma_3) = 0$. That is,

$$C_1(X) = -(\sigma_1 - \sigma_2)^{-1}(\sigma_1 - \sigma_3)^{-1} \cdot (X - \sigma_2)(X - \sigma_3).$$

- $B_2(\sigma_2) = 1$ and $B_2(\sigma_1) = B_2(\sigma_3) = 0$. That is, $B_2(X)$ is the same polynomial as $A_1(X)$.

- $C_3(\sigma_1) = 0, C_3(\sigma_2) = 1,$ and $C_3(\sigma_3) = -1$. That is,

$$C_3(X) = (X - \sigma_1) \cdot (X - \sigma_3) \cdot (\sigma_2 - \sigma_1)^{-1} \cdot (\sigma_2 - \sigma_3)^{-1} - (\sigma_3 - \sigma_1)^{-1} \cdot (\sigma_3 - \sigma_2)^{-1} \cdot (X - \sigma_1) \cdot (X - \sigma_2).$$

- $C'(\sigma_1) = 4, C'(\sigma_2) = 0,$ and $C'(\sigma_3) = 9$. That is,

$$C'(X) = 4(X - \sigma_2) \cdot (X - \sigma_3) \cdot (\sigma_1 - \sigma_2)^{-1} \cdot (\sigma_1 - \sigma_3)^{-1} + 9(\sigma_3 - \sigma_1)^{-1} \cdot (\sigma_3 - \sigma_2)^{-1} \cdot (X - \sigma_1) \cdot (X - \sigma_2).$$

- The following polynomials are identically zero: $A_2, A_3, B_1, B_3, C_2, A',$ and B' .

Turning W into a polynomial that vanishes on H if and only if W is a correct transcript. Let $g_{x,y,W}(t)$ denote the following univariate polynomial:

$$\left(\left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot A_i(t) \right) + A'(t) \right) \cdot \left(\left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot B_i(t) \right) + B'(t) \right) - \left(\left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot C_i(t) \right) + C'(t) \right) \quad (14.7)$$

By design, $g_{x,y,W}$ vanishes on H if and only if all constraints are satisfied, i.e., if and only if W is a correct transcript for $\{\mathcal{C}, x, y\}$. To check whether $g_{x,y,W}$ vanishes on H , we rely on Lemma 8.3, which also played a key role in our constructions of efficient PCPs and IOPs (Chapters 8 and 9) and is restated here for the reader's convenience.

Lemma 8.3. (*Ben-Sasson and Sudan [BS08]*) *A degree d univariate polynomial $g_{x,y,z}(z)$ vanishes on H if and only if the polynomial $\mathbb{Z}_H(z) := \prod_{\alpha \in H} (z - \alpha)$ divides $g_{x,y,z}(z)$, i.e., iff there exists a polynomial h^* with $\deg(h^*) \leq d - |H|$ such that $g_{x,y,z}(z) = \mathbb{Z}_H(z) \cdot h^*(z)$.*

By inspection, the degree of the polynomial $g_{x,y,W}$ is at most $d = 2(\ell - 1)$, where $\ell = |S| + |y| - |w|$ is the number of constraints. By Lemma 8.3, to convince \mathcal{V} that $g_{x,y,W}$ vanishes on H , the proof merely needs to convince \mathcal{V} that $g_{x,y,W}(z) = \mathbb{Z}_H(z) \cdot h^*(z)$ for some polynomial h^* of degree $d - |H| = \ell - 1$. To be convinced of this, \mathcal{V} can pick a random point $r \in \mathbb{F}$ and check that

$$g_{x,y,W}(r) = \mathbb{Z}_H(r) \cdot h^*(r). \quad (14.8)$$

Indeed, because any two distinct degree $(\ell - 1)$ polynomials can agree on at most $d + 1$ points, if $g_{x,y,W} \neq \mathbb{Z}_H \cdot h^*$, then this equality will fail with probability at least $1 - (\ell - 1)/|\mathbb{F}|$.

To this end, a correct proof represents two linear functions. The first is $f_{\text{coeff}(h^*)}$, where $\text{coeff}(h^*)$ denotes the vector of coefficients of h^* (recall that f_v for any vector $v \in \mathbb{F}^\ell$ denotes the ℓ -variate linear function $f_v(x) := \langle v, x \rangle$). The second is f_W . Note that

$$f_{\text{coeff}(h^*)}(1, r, r^2, \dots, r^{\ell-1}) = h^*(r), \quad (14.9)$$

so \mathcal{V} can evaluate $h^*(r)$ with a single query to the proof. Similarly, \mathcal{V} can evaluate $g_{x,y,W}$ at r by evaluating f_W at the three vectors $(A_1(r), \dots, A_S(r))$, $(B_1(r), \dots, B_S(r))$, and $(C_1(r), \dots, C_S(r))$.

Just as in the linear PCP of Section 14.3, the verifier also has to perform linearity testing on $f_{\text{coeff}(h^*)}$ and f_W . The verifier must also replace the four queries described above with two queries each to ensure that all queries are uniformly distributed. These complications arise because we required that a linear PCP be sound against proofs that are *non-linear* functions of queries. We remark that for purposes of the non-interactive argument system of the next section (Section 14.5), the linear PCP verifier need not perform linearity testing nor ensure that any of its queries are uniformly distributed. This is because the cryptographic techniques in that section bind the argument system to *linear functions*, so the underlying information-theoretic protocol does not need to bother testing whether the function is in fact linear. In contrast, the cryptographic techniques of Section 14.2 only bind the prover to some function which was not necessarily linear, hence the need for the underlying linear PCP to be sound against proofs that are non-linear functions.

Protocol Costs. The costs of the argument system obtained by combining QAPs with the commitment protocol for linear functions (Section 14.2) are summarized in Table 14.2. The honest prover \mathcal{P} needs to perform the following steps, assuming \mathcal{P} knows a witness w for \mathcal{C} . First, evaluate \mathcal{C} gate-by-gate to find a correct transcript W . Second, compute the polynomial $g_{x,y,W}(t)$. Third, divide $g_{x,y,W}$ by \mathbb{Z}_H to find the quotient polynomial h^* . Fourth run the linear commitment/reveal protocol described in Section 14.2, to commit to $f_{\text{coeff}(h^*)}$ and f_W and answer the verifier's queries.

The first and fourth steps can clearly be done in time $O(S)$. The second step can be done in time $O(S \log^2 S)$ using standard multipoint interpolation algorithms based on the Fast Fourier Transform (FFT). The third step can be done in time $O(S \log S)$ using FFT-based polynomial division algorithms.

Total communication from \mathcal{V} to \mathcal{P} is $\Theta(S)$ as well (this cost can also be amortized), but the communication in the reverse direction is just a constant number of field elements per input. \mathcal{V} 's time and \mathcal{P} 's time are both $\tilde{\Theta}(S)$, but if \mathcal{V} is simultaneously verifying \mathcal{C} 's execution over a large batch of inputs, then the $\Theta(S)$ cost for \mathcal{V} can be amortized over the entire batch.

Note that the verifier's check does require \mathcal{V} to evaluate $\mathbb{Z}_H(r)$, where $\mathbb{Z}_H(X) = \prod_{\sigma \in H} (X - \sigma)$ is the vanishing polynomial of H . Since the verifier requires $O(S)$ time just to specify the linear PCP queries,

$\mathcal{V} \rightarrow \mathcal{P}$ Communication	$\mathcal{P} \rightarrow \mathcal{V}$ Communication	Queries	\mathcal{V} time	\mathcal{P} time
$O(S)$ field elements	$O(1)$ field elements	$O(1)$	$\tilde{O}(S)$	$\tilde{O}(S)$

Table 14.2: Costs of the argument system from Section 14.4 when run on a circuit satisfiability instance of size S . The \tilde{O} notation hides polylogarithmic factors in S . Note that the verifier’s cost and the communication cost can be amortized when outsourcing the circuit \mathcal{C} ’s execution on a *batch* of inputs. The stated bound on \mathcal{P} ’s time assumes \mathcal{P} knows a witness w for \mathcal{C} .

\mathcal{V} ’s asymptotic time bound in the linear PCP (as well as in the argument system resulting from applying the transformation of Section 14.2 to the linear PCP) is not affected by computing $\mathbb{Z}_H(r)$ directly via S subtractions and multiplications. Nonetheless, H could be chosen carefully to ensure that $\mathbb{Z}_H(x)$ is sparse, thereby enabling $\mathbb{Z}_H(r)$ to be evaluated in $o(S)$ time. For example, if H is chosen to be a multiplicative subgroup of \mathbb{F} of order ℓ (the setting considered in Section 9.3, see Equation (9.2)), then $\mathbb{Z}_H(X) = X^\ell - 1$. Clearly, then $\mathbb{Z}_H(r)$ can be evaluated with $O(\log n)$ field multiplications.

In the non-interactive argument presented in the next section, $\mathbb{Z}_H(r)$ will simply be provided to the verifier as part of the trusted setup procedure, which takes time $O(S)$ regardless of whether or not H is chosen to ensure \mathbb{Z}_H is a sparse polynomial.

14.5 Non-Interactivity and Public Verifiability

14.5.1 Informal Overview

We have already seen (Section 14.2) how to convert the linear PCP of the previous section to a succinct interactive argument using any additively-homomorphic encryption scheme. We cannot apply the Fiat-Shamir transformation to render this argument system non-interactive because it is not public coin—the argument system makes use of an additively homomorphic encryption scheme for which the verifier chooses the private key, and if the prover learns the private key it can break soundness of the argument system.

Instead, linear PCPs can be converted to non-interactive arguments using pairing-based techniques extremely similar to the pairing-based polynomial commitment of Section 13.2.2. The idea is as follows (we simplify slightly in this informal overview, deferring a complete description of the protocol until Section 14.5.3).¹⁴⁸ Rather than having the verifier send the linear PCP queries to the prover “in the clear” as in the interactive argument of Section 14.2, the linear PCP queries $q^{(1)}, \dots, q^{(k)}$, will be *encoded* in the exponent of a group generator g for pairing-friendly group \mathbb{G} , and the encodings provided to the prover via inclusion in a structured reference string. The argument system then exploits the additive homomorphism of the encoding (i.e., the encoding g^{x+y} of $x+y \in \mathbb{F}_p$ equals the product of the encodings g^x, g^y of x and y individually, so long as $|\mathbb{G}| = p$). If $\pi(x) = \sum_j c_j x_j$ denotes a linear PCP proof, the additive homomorphism allows the prover to evaluate the encoding of $\pi(q^{(1)}), \dots, \pi(q^{(k)})$ and send them to the verifier. Finally, the argument system verifier accepts if and only if the linear PCP verifier would accept the responses $\pi(q^{(1)}), \dots, \pi(q^{(k)})$. Since the argument system prover did not send $\pi(q^{(1)}), \dots, \pi(q^{(k)})$ in the clear, but rather *encodings* $g^{\pi(q^{(1)})}, \dots, g^{\pi(q^{(k)})}$, it is not immediately obvious how the argument system verifier can make this determination. This is where pairings come in.

Observe that the verifier’s check in the linear PCP is a total-degree-2 function of the responses to the PCP queries, and recall from Section 13.2.1 that the entire point of pairings is that they allow for a single

¹⁴⁸In this section, we use the serif font g rather than g to denote a generator of a pairing-friendly group \mathbb{G} , to distinguish the group generator from the polynomial $g_{x,y,W}$ defined in the previous section.

“multiplication check” to be performed on encoded values, without the need to decode the values. This enables the argument system verifier to perform the linear PCP verifier’s check “in the exponent”. Indeed, recall that the linear PCP verifier checks that $g_{x,y,W}(r) = \mathbb{Z}_H(r) \cdot h^*(r)$. Letting

$$q^{(1)} = (A_1(r), \dots, A_S(r)),$$

$$q^{(2)} = (B_1(r), \dots, B_S(r)),$$

and

$$q^{(3)} = (C_1(r), \dots, C_S(r)),$$

then

$$g_{x,y,W}(r) = \left(f_W(q^{(1)}) + A'(r) \right) \cdot \left(f_W(q^{(2)}) + B'(r) \right) - \left(f_W(q^{(3)}) + C'(r) \right),$$

which is clearly a function of total degree two in the linear PCP prover responses $f_W(q^{(1)})$, $f_W(q^{(2)})$, and $f_W(q^{(3)})$. Similarly, letting $q^{(4)} = (1, r, \dots, r^S)$, the right hand side of the verifier’s check is a linear (degree 1) function of the linear PCP prover response $f_{\text{coeff}(h^*)}(q^{(4)})$.

14.5.2 A Complication: Linear Interactive Proofs vs. Linear PCPs

The argument system sketched above runs into the following complication. While (under appropriate Knowledge of Exponent assumptions) the pairing-based cryptography forces the argument system prover to answer each encoded linear PCP query in a manner consistent with a linear function, it does not ensure that all queries are answered with *the same* linear function.¹⁴⁹ That is, for the argument system to be sound, we really need the underlying linear PCP to be sound against provers that use a different linear function to answer each query.¹⁵⁰ Such a linear PCP is called a (2-message) *linear interactive proof* (LIP) [BCI⁺13].

Bitansky et al. [BCI⁺13] give a simple and efficient method for translating any linear PCP into a LIP. Specifically, if soundness of the linear PCP requires that queries $q^{(1)}, \dots, q^{(k')}$ be answered with the same linear function, the LIP verifier simply adds an extra query $q^{(k'+1)} = \sum_{i=1}^{k'} \beta_i q^{(i)}$ to the linear PCP, where $\beta_1, \dots, \beta_{k'}$ are randomly chosen field elements known only to the verifier. That is, $q^{(k'+1)}$ is a random linear combination of the relevant linear PCP queries. The LIP verifier checks that the answer $a_{k'+1}$ to the $(k+1)$ ’st query equals $\sum_{i=1}^{k'} \beta_i a_i$, and if so, feeds answers a_1, \dots, a_k to the linear PCP verifier. It can be shown that if the linear PCP is complete and knowledge-sound, then the resulting LIP is as well. We omit the proof of this fact, but the idea is to argue that if the LIP prover does not answer all $k' + 1$ queries $q^{(1)}, \dots, q^{(k')}, q^{(k'+1)}$ using the same linear function for each query, then there is some nonzero linear function π such that the prover will pass the LIP verifier’s final check only if $\pi(\beta_1, \dots, \beta_{k'}) = 0$. Since $\beta_1, \dots, \beta_{k'}$ are chosen uniformly at random from \mathbb{F} , the Schwartz-Zippel lemma (Lemma 3.3) implies that this occurs with probability at most $1/|\mathbb{F}|$.

14.5.3 Complete Description of the SNARK

Here is the entire non-interactive argument system. Recall that $q^{(1)}, \dots, q^{(4)}$ were defined in Section 14.5.1. Accounting for the transformation from a linear PCP to an LIP of Section 14.5.2, we define a 5th query vector $q^{(5)} := \sum_{i=1}^3 \beta_i q^{(i)}$, where β_1, \dots, β_3 are randomly chosen elements of \mathbb{F}_p . (We do not include the

¹⁴⁹In fact, the cryptography does not prevent the prover from answering the i th (encoded) query $q^{(i)}$ with (an encoding of) a linear combination of entries of *all of the queries* $q^{(1)}, \dots, q^{(3)}$.

¹⁵⁰More precisely, owing to Footnote 149, the linear PCP needs to be secure against provers that answer each of the four queries with different linear function of all four queries.

4th query in this random linear combination because soundness of the linear PCP from Section 14.4 only requires that the first 3 queries be answered with the same linear function f_W , and completeness in fact requires that the 4th query be answered with a different linear function, namely $f_{\text{coeff}(h^*)}$.)

For every entry $q_j^{(i)}$ of each of the five LIP queries $q^{(1)}, \dots, q^{(5)}$, the SRS contains the pair $(g^{q_j^{(i)}}, g^{\alpha q_j^{(i)}})$ where α is chosen at random from $\{1, \dots, p-1\}$. The verification key (i.e., the information provided to the verifier by the trusted setup procedure) contains the quantities $g, g^\alpha, g^{A'(r)}, g^{B'(r)}, g^{C'(r)}, g^{\mathbb{Z}_H(r)}, g^{\beta_1}, g^{\beta_2}, g^{\beta_3}$. Note that all quantities in the SRS can be computed during the setup phase because they depend only on the circuit \mathcal{C} whose satisfiability is being verified, and not on the public input x or witness w for \mathcal{C} . Using the SRS and additive homomorphism, the prover computes and sends to the verifier five pairs of group elements $(g_1, g'_1), \dots, (g_4, g'_4), (g_5, g'_5)$ claimed to equal

$$\begin{aligned} & (g^{f_W(q^{(1)})}, g^{\alpha \cdot f_W(q^{(1)})}), \\ & (g^{f_W(q^{(2)})}, g^{\alpha \cdot f_W(q^{(2)})}), \\ & (g^{f_W(q^{(3)})}, g^{\alpha \cdot f_W(q^{(3)})}), \\ & (g^{f_{\text{coeff}(h^*)}(q^{(4)})}, g^{\alpha \cdot f_{\text{coeff}(h^*)}(q^{(4)})}), \end{aligned}$$

and

$$(g^{f_W(q^{(5)})}, g^{\alpha \cdot f_W(q^{(5)})}).$$

The verifier performs the following checks. First, it checks that

$$e(g_1 \cdot g^{A'(r)}, g_2 \cdot g^{B'(r)}) = e(g_3 \cdot g^{C'(r)}, g) \cdot e(g^{\mathbb{Z}_H(r)}, g_4). \quad (14.10)$$

Second, it checks that

$$\prod_{i=1}^3 e(g^{\beta_i}, g_i) = e(g_5, g). \quad (14.11)$$

Third, for each of the five pairs (g_i, g'_i) for $i = 1, \dots, 5$, the verifier checks that

$$e(g_i, g^\alpha) = e(g, g'_i). \quad (14.12)$$

14.5.4 Establishing Completeness and Knowledge Soundness

Completeness of the SNARK holds by design. Indeed, by bilinearity of e , the first check of the verifier (Equation (14.10)) is specifically designed to pass if $g_{x,y,W}(r) = \mathbb{Z}_H(r) \cdot h^*(r)$ and the prover returns the prescribed proof elements. The second check (Equation (14.11)) will pass if and only if $g_5 = \prod_{i=1}^3 g_i^{\beta_i}$, which will be the case if the prover behaves as prescribed. Similarly, the final set of checks (Equation (14.12)) will pass if indeed $g'_i = g_i^\alpha$ for all i .

The proof of knowledge soundness relies on the following two cryptographic assumptions. These are mild variants of the two assumptions (PKoE and SDH) that we relied upon for the pairing-based polynomial commitment of Section 13.2.2.

Knowledge of Exponent Assumption (KEA). This is a variant of the PKoE assumption. Recall that the SRS for the SNARK of this section consists of $t = O(S)$ many pairs of the form (g_i, g_i^α) for $i = 1, \dots, t$. The Knowledge of Exponent assumption essentially guarantees that for any polynomial-time algorithm that is given such an SRS as input and is capable of outputting pairs (f, f') such that $f' = f^\alpha$, there is an efficient extractor algorithm that outputs coefficients c_1, \dots, c_t explaining (f, f') , in the sense that $f = \prod_{i=1}^t g_i^{c_i}$. See Section 13.2.2 for discussion of the intuition behind such an assumption and why it is reasonable.

Poly-Power Discrete Logarithm is Hard. This assumption posits that, if r is chosen at random from \mathbb{F}_p , then any polynomial time algorithm, when given as input the encodings of $t \leq \text{poly}(S)$ many powers of r (i.e., $g, g^r, g^{r^2}, \dots, g^{r^t}$), is incapable of outputting r except with negligible probability.

Informally, the final set of five checks the SNARK verifier performs (Equation (14.12)) guarantees by KEA that the SNARK prover answers all of the LIP queries using linear functions, and in fact the prover “knows” these linear functions. To clarify, since in the SNARK the LIP queries are encoded in the exponent of g , the SNARK prover is applying the linear function to the exponents, by taking products of constant powers of the encoded query entries in the SRS.

The remaining two checks that the SNARK verifier performs ensures that these linear functions would convince the verifier to accept in the LIP obtained by applying the transformation of Section 14.5.2 to the linear PCP of Section 14.4. Knowledge soundness of the SNARK then follows from that of the LIP.

In more detail, the analysis establishing that the SNARK is knowledge sound shows how to transform any argument system prover that convinces the argument system verifier to accept with non-negligible probability into either a witness w for the claim that $\mathcal{C}(x, w) = y$, or a polynomial time algorithm \mathcal{A} that breaks the poly-power discrete logarithm assumption. Because the SNARK prover passes the final set of 5 checks performed by the verifier (Equation (14.12)) with non-negligible probability, the KEA implies that there is an efficient extractor \mathcal{E} outputting linear functions $\pi_1, \dots, \pi_5: \mathbb{F}^t \rightarrow \mathbb{F}$ that “explain” the query responses as linear combinations (in the exponent) of SRS elements. That is, for $i = 1, \dots, 5$, if the SRS σ consists of pairs of group elements (f_j, f_j^α) for $j = 1, \dots, |\sigma|$, let $c_{i,1}, \dots, c_{i,|\sigma|}$ denote the coefficients of π_i . Then for $i \in \{1, 2, 3, 4, 5\}$,

$$g_i = \prod_{j=1}^{|\sigma|} f_j^{c_{i,j}}.$$

For notational convenience, let us write π_1 as f_W , and π_4 as $f_{\text{coeff}(h^*)}$. Let $g_{x,y,W}$ and h^* be the polynomials implied by W and h^* via Equations (14.7) and (14.9). The argument system’s verifier’s first and second checks ensure that these linear functions convince the LIP verifier to accept with non-negligible probability. In particular, the LIP soundness analysis then implies that $\pi_1 = \pi_2 = \pi_3 = f_W$ and hence that $g_{x,y,W}(r) = \mathbb{Z}_H(r) \cdot h^*(r)$.

If $g_{x,y,W} = \mathbb{Z}_H \cdot h^*$, then W is a valid circuit transcript for the claim that there exists a w such that $\mathcal{C}(x, w) = y$, so to prove knowledge soundness it suffices to suppose that $g_{x,y,W} \neq \mathbb{Z}_H \cdot h^*$ and show that this would contradict the poly-power discrete logarithm assumption.

If $g_{x,y,W} \neq \mathbb{Z}_H \cdot h^*$, then since both the left hand side and right hand side are polynomials of degree most 2ℓ where $\ell = S + |y| - |w|$, there are at most 2ℓ points r' for which $g_{x,y,W}(r') = \mathbb{Z}_H(r') \cdot h^*(r')$, and all such points r' can be enumerated in $\text{poly}(S)$ time using a polynomial factorization algorithm. Consider the algorithm \mathcal{A} that selects one of these points r' at random. Clearly \mathcal{A} runs in polynomial time, and with non-negligible probability (at least $1/(2\ell)$), it outputs r . We claim that this violates the poly-power discrete logarithm assumption. Indeed, since $A_1, \dots, A_S, B_1, \dots, B_S, C_1, \dots, C_S, A', B'$, and C' are all polynomials of degree at most ℓ that are all computable in $\text{poly}(S)$ time, the SRS for the SNARK of this section consists

entirely of encodings of known linear combinations of powers-of- r (i.e., of products of known powers of $g, g^r, g^{r^2}, \dots, g^{r^\ell}$), plus additional group elements equal to these values raised to either $\alpha, \beta_1, \beta_2, \beta_3, \alpha \cdot \beta_1, \alpha \cdot \beta_2, \text{ or } \alpha \cdot \beta_3$, where $\alpha, \beta_1, \beta_2, \beta_3$ are uniform random elements of $\{1, \dots, p-1\}$. Hence, a string distributed identically to the entire SRS of the SNARK (which \mathcal{A} is given access to) can be computed in polynomial time given the input encodings referred to in the poly-power discrete logarithm assumption. Since \mathcal{A} outputs r with non-negligible probability, \mathcal{A} violates the assumption.

14.5.5 Achieving Zero-Knowledge

In the SNARK above, the proof contains encodings of f_W evaluated at various points, where W is a valid circuit transcript. This leaks information to the verifier that the verifier cannot compute on its own, since it does not know W . Hence, the SNARK is not zero-knowledge. To render the protocol zero-knowledge, we modify the underlying LIP to be honest-verifier zero-knowledge. This ensures that the resulting SNARK is zero-knowledge even against dishonest verifiers, by the following reasoning. Because the SNARK verifier does not send any message to the prover, honest-verifier and malicious-verifier zero-knowledge are equivalent for the SNARK. The SNARK verifier only sees the verification key (which is generated in polynomial time and is independent of the witness), and encodings of the LIP prover's responses to the LIP verifier's queries. Once the proving and verification keys are generated, these encodings are deterministic, efficiently computable functions of the responses. Hence, since the LIP is honest-verifier perfect zero-knowledge, so is the resulting SNARK. That is, the simulator for the SNARK verifier's view simply runs the simulator for the LIP verifier's view, and outputs encodings of the LIP prover's messages instead of the messages themselves.

Recall that in the nonzero-knowledge LIP, the prover established that $g_{x,y,W} = \mathbb{Z}_H \cdot h^*$ for a valid transcript W , where

$$g_{x,y,W}(t) = \left(\left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot A_i(t) \right) + A'(t) \right) \cdot \left(\left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot B_i(t) \right) + B'(t) \right) - \left(\left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot C_i(t) \right) + C'(t) \right).$$

This required the LIP verifier to pick a random $r \in \mathbb{F}$ and obtain from the prover the following four evaluations:

$$\begin{aligned} & h^*(r), \\ & \sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot A_i(r), \\ & \sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot B_i(r), \end{aligned}$$

and

$$\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot C_i(r).$$

These four values leak information to the LIP verifier, who cannot efficiently compute W or h^* .

To render the LIP zero-knowledge, the prover picks three random values $r_A, r_B, r_C \in \mathbb{F}$, and considers a "perturbed" version $g'_{x,y,W}$ of $g_{x,y,W}$, in which each constituent function comprising $g_{x,y,W}$ has added to it a random multiple of the vanishing polynomial \mathbb{Z}_H of H . Specifically, letting

$$A(t) := \left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot A_i(t) \right) + A'(t),$$

$$B(t) := \left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot B_i(t) \right) + B'(t),$$

$$C(t) := \left(\sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot C_i(t) \right) + C'(t),$$

define:

$$g'_{x,y,W}(t) := (A(t) + r_A \mathbb{Z}_H(t)) \cdot (B(t) + r_B \mathbb{Z}_H(t)) - (C(t) + r_C \mathbb{Z}_H(t)). \quad (14.13)$$

Note that $g'_{x,y,W}(t) = g_{x,y,W}(t) + r_B \mathbb{Z}_H(t)A(t) + r_A \mathbb{Z}_H(t)B(t) + r_A r_B (\mathbb{Z}_H(t))^2 - r_C \mathbb{Z}_H(t)$. Just as $g_{x,y,W}$ vanished on H if and only if W is a valid transcript, the same can be said for $g'_{x,y,W}$, because the “added factors” in $g'_{x,y,W}$ are multiples of the polynomial \mathbb{Z}_H , which vanishes on H .

To prove that $g'_{x,y,W}$ vanishes on H , it is sufficient for the prover to establish that there exists a polynomial h' such that $g'_{x,y,W} = h' \cdot \mathbb{Z}_H$. Note that this is satisfied by

$$h' = h^* + r_B \cdot A(t) + r_A \cdot B(t) + r_A r_B \mathbb{Z}_H - r_C.$$

The LIP verifier can (with soundness error at most the $2S/(|\mathbb{F}| - S)$) check that this equality of formal polynomials holds by confirming that the right hand and left hand sides agree at a random point $r \in \mathbb{F} \setminus H$.

The zero-knowledge LIP proof consists of two linear functions. The first is claimed to equal $f_{\text{coeff}(h')}$, defined as usual so that $f_{\text{coeff}(h')}(1, r, r^2, \dots, r^{\deg(h')}) = h'(r)$. The second is prescribed to equal $f_{W'}$ where W' is the vector $W \circ r_A \circ r_B \circ r_C \in \mathbb{F}^{S+3}$. That is, W' is the valid transcript W , with the random values r_A, r_B, r_C that are chosen by the prover appended.

The honest verifier will query $f_{W'}$ at three locations:

$$q^{(1)} = (A_1(r), \dots, A_S(r), \mathbb{Z}_H(r), 0, 0),$$

$$q^{(2)} = (B_1(r), \dots, B_S(r), 0, \mathbb{Z}_H(r), 0),$$

and

$$q^{(3)} = (C_1(r), \dots, C_S(r), 0, 0, \mathbb{Z}_H(r))$$

to obtain the three values:

$$v_1 := r_A \cdot \mathbb{Z}_H(r) + \sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot A_i(r),$$

$$v_2 := r_B \cdot \mathbb{Z}_H(r) + \sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot B_i(r),$$

and

$$v_3 := r_C \cdot \mathbb{Z}_H(r) + \sum_{\text{gates } i \text{ in } \mathcal{C}} W_i \cdot C_i(r).$$

The honest LIP verifier will then pick a point $r \in \mathbb{F} \setminus H$ at random and query the $f_{\text{coeff}(h')}$ at a single point $q^{(4)} := (1, r, r^2, \dots, r^S)$ to obtain a value v_4 claimed to equal $h'(r)$. Then the verifier will check that

$$(v_1 + A'(r)) \cdot (v_2 + B'(r)) - (v_3 + C'(r)) = v_4 \cdot \mathbb{Z}_H(r).$$

Finally, following Section 14.5.2, in order to confirm that the LIP prover answered the queries $q^{(1)}$, $q^{(2)}$, and $q^{(3)}$ with the same linear function, the verifier will also choose $\beta_1, \beta_2, \beta_3$ at random from \mathbb{F} and query $f_{W'}$ at location $q^{(5)} = \sum_{i=1}^3 \beta_i q^{(i)}$ to obtain response v_5 , and check that $\sum_{i=1}^3 \beta_i v_i = v_5$. By the discussion in Section 14.5.2, if the LIP prover passes this check, then with high probability a single linear function $f_{W'}$ was used to answer $q^{(1)}$, $q^{(2)}$, and $q^{(3)}$.

Analysis of the LIP. Completeness of this LIP holds by design. Soundness holds because for any linear functions $f_{\text{coeff}(h')}$ and $f_{W'}$ that cause the LIP verifier to accept, then $\text{coeff}h'$ must specify the coefficients of a polynomial h' and W' must specify a witness $W \in \mathbb{F}^S$ followed by three values r_A, r_B, r_C such that $h'(r) \cdot \mathbb{Z}_H(r) = g'_{x,y,W}(r)$, where $g'_{x,y,W}(t)$ is as defined in Equation (14.13). This implies that W is a correct witness.

The LIP is honest-verifier zero-knowledge by the following reasoning. Since $r \notin H$, we may conclude that $\mathbb{Z}_H(r) = \prod_{a \in H} (r - a) \neq 0$. Combined with the fact that r_A, r_B , and r_C are independent, uniform random field elements, it follows that $\mathbb{Z}_H(r) \cdot r_A, \mathbb{Z}_H(r) \cdot r_B$, and $\mathbb{Z}_H(r) \cdot r_C$ are uniform random field elements as well. Hence, $f_{W'}(q^{(1)})$, $f_{W'}(q^{(2)})$, and $f_{W'}(q^{(3)})$ are themselves uniform random field elements, as each is some fixed quantity plus a uniform random field element (e.g., $f_{W'}(q^{(1)}) = f_W(A_1(r), \dots, A_S(r)) + \mathbb{Z}_H(r) \cdot r_A$).

Meanwhile, for any choice of $r \in \mathbb{F}$, $v_4 = h'(r)$ is always equal to

$$\left((v_1 + A'(r)) \cdot (v_2 + B'(r)) - (v_3 + C'(r)) \right) \mathbb{Z}_H(r)^{-1}. \quad (14.14)$$

Hence, the simulator can choose r at random from \mathbb{F} , and simply set v_1, v_2, v_3 (i.e., the simulated responses to queries $q^{(1)}, q^{(2)}, q^{(3)}$) to be uniform random field elements, and then set v_4 as per Equation (14.14). Finally, the simulator chooses $\beta_1, \beta_2, \beta_3$ at random from \mathbb{F} , and computes the simulated response v_5 to $q^{(5)}$ as $\sum_{i=1}^3 \beta_i v_i$. This is a perfect simulation of the LIP verifier's view.

Historical notes. The zk-SNARK described above is nearly identical to the one for QAPs given in [GGPR13]. Minor differences arise in our treatment, stemming from our use of the linear-PCP-to-LIP from subsequent work [BCI⁺13] in the construction and analysis of the SNARK presented here. [PHGR13] provided concrete improvements to the zk-SNARK from [GGPR13], and implemented the resulting variant. Other optimized variants were presented and implemented in [BCG⁺13, BCTV14b]. A nice, self-contained presentation of one of these variants can be found in [BCTV14b, Appendix B]. Soundness proofs for SNARKs derived from linear PCPs can apparently be somewhat subtle and/or error-prone. For example, Gabizon [Gab19b] identified a subtle security flaw in an earlier version of the variant from [BCTV14b], owing to unnecessary group elements accidentally being included in the SRS.¹⁵¹ Gabizon proved that when the group elements are omitted from the SRS, the variant SNARK is sound in the generic group model.

Groth's SNARK. Groth [Gro16] gave an influential variant of the zk-SNARK of this section, in which the proof consists of only 3 group elements, and proved his SNARK to be knowledge sound in the generic group model. Fuchsbauer, Kiltz, and Loss [FKL18] extended the security proof to the Algebraic Group Model.

¹⁵¹The unnecessary elements were encodings $(g^{x_i}, g^{\alpha x_i})$ of entries of the public input x . Since the prover knows x , these encodings turn out to enable any prover to take a valid proof of knowledge of a witness w such that $\mathcal{C}(x, w) = y$, and for any $x' \neq x$ turn it into a “proof” for the potentially invalid statement that $\mathcal{C}(x', w) = y$.

Chapter 15

SNARK Composition and Recursion

15.1 Composing Two Different SNARKs

Consider two SNARK systems, \mathcal{I} and \mathcal{O} , say for arithmetic circuit-satisfiability, with different cost profiles. The prover in \mathcal{I} is very fast (say, linear in the size of the statement being proven), but the proofs and verification time are fairly large (though still sublinear in the size of the statement being proven, e.g., square root of the circuit size). In contrast, the prover in \mathcal{O} is slower (say, superlinear in the size of the circuit by logarithmic factors, and with a large leading constant factor), but the proofs and verification time are very short and fast (say, of length logarithmic or even constant in the circuit size). Is it possible to combine them to get the best of both worlds? That is, we seek a SNARK \mathcal{F} with the fast prover speed of \mathcal{I} and the short proof length and fast verification of \mathcal{O} .

The answer is yes, at least in principle, via a technique called proof composition. This works as follows. Suppose the \mathcal{F} -prover $\mathcal{P}_{\mathcal{F}}$ claims to know a witness w such that $\mathcal{C}(w) = 1$, where \mathcal{C} is a specified circuit. $\mathcal{P}_{\mathcal{F}}$ can use \mathcal{I} to generate a SNARK proof π of the claim at hand. But since π is pretty big and verifying it is somewhat slow, $\mathcal{P}_{\mathcal{F}}$ doesn't want to explicitly send π to the \mathcal{F} verifier. Rather, $\mathcal{P}_{\mathcal{F}}$ can use the \mathcal{O} -SNARK system to *prove* to the \mathcal{F} -verifier that it knows π . It is this \mathcal{O} -proof π' that $\mathcal{P}_{\mathcal{F}}$ actually sends to the verifier. Put another way, $\mathcal{P}_{\mathcal{F}}$ uses the fast-verification SNARK \mathcal{O} to establish knowledge of an \mathcal{I} -proof π that would have convinced the \mathcal{I} verifier that $\mathcal{P}_{\mathcal{F}}$ knows a w such that $\mathcal{C}(w) = 1$.

The above procedure requires taking the verification procedure of \mathcal{I} and feeding it through the proof machinery of \mathcal{O} . That is, the \mathcal{I} -verifier must be represented as an arithmetic circuit \mathcal{C}' and the \mathcal{O} prover then applied to \mathcal{C}' to establish knowledge of a π such that $\mathcal{C}'(\pi) = 1$.¹⁵²

Let $\mathcal{F} = \mathcal{O} \circ \mathcal{I}$ denote the above composed proof system. Here, \mathcal{O} stands for the “outer” SNARK and \mathcal{I} stands for the “inner” SNARK. The motivation for this terminology is that one thinks of the \mathcal{O} -proof π' that is actually sent to the verifier in \mathcal{F} as having an \mathcal{I} -proof π “living inside of it”. That is, the \mathcal{O} -proof π' attests that whoever generated the proof knows some \mathcal{I} -proof π for the claim at hand.

Costs of the composed proof system. The final proof length and verification time of \mathcal{F} is the size of the proof generated by \mathcal{O} applied to the \mathcal{I} verifier circuit \mathcal{C}' . Since the \mathcal{O} -proof and verification procedure are short and fast, so is the \mathcal{F} -proof and verification procedure.

¹⁵²There is nothing special about circuit-satisfiability in this example. What matters is that the verification procedure of \mathcal{I} be represented in whatever format \mathcal{O} requires to allow \mathcal{P} to establish that it knows an \mathcal{I} -proof π that would have caused the \mathcal{I} verifier to accept. See Chapter 5 for additional discussion of intermediate representations other than circuits, including RICS.

The \mathcal{F} prover first has to generate the \mathcal{I} -proof π for \mathcal{C} (which is by assumption fast), and then has to generate the \mathcal{O} -proof for \mathcal{C}' . While the \mathcal{O} prover is slow, the key point is that \mathcal{C}' should be *much smaller* than \mathcal{C} , since the verification procedure of \mathcal{I} is sublinear (e.g., square root) in the size of \mathcal{C} . Hence, the time required by the \mathcal{F} prover to generate the \mathcal{I} -proof that $\mathcal{C}'(\pi) = 1$ should be dwarfed by the time required to compute π in the first place. Hence, the \mathcal{F} prover time is extremely close to that of the \mathcal{I} prover, which by assumption is fast. The best of both worlds has been achieved.

There are other potential benefits of proof composition beyond reducing verification costs. For example, if the inner SNARK \mathcal{I} is not zero-knowledge, but the outer SNARK \mathcal{O} is zero-knowledge, the composed SNARK \mathcal{F} will be zero-knowledge. Hence, composition can be used to transform a highly efficient but non-zero-knowledge SNARK \mathcal{I} into a new SNARK $\mathcal{O} \circ \mathcal{I}$ that is zero-knowledge.

15.2 Deeper Compositions of SNARKs

As in the previous section, imagine a SNARK \mathcal{I} for circuit satisfiability instances of size S in which the verification procedure, when itself represented as an arithmetic circuit, has size $O(S^{1/2})$, and proofs have size $O(S^{1/2})$ as well. That is, verification is sublinear relative to the cost of evaluating the circuit gate-by-gate on a witness w , but is still more expensive than we might like. In principle, self-composition can be used to obtain a SNARK with lower verification cost.

Composing \mathcal{I} with *itself* yields a new SNARK $\mathcal{F} = \mathcal{I} \circ \mathcal{I}$ with proof size and verification time $O((S^{1/2})^{1/2}) = O(S^{1/4})$. One more invocation of composition, say with \mathcal{F} as the outer SNARK and with \mathcal{I} as the inner SNARK, yields yet another SNARK, now with verification time $O((S^{1/4})^{1/2}) = S^{1/8}$. In this way, the more invocations of composition, the smaller the proofs and faster the verification time of the resulting SNARK. One can fruitfully continue this process until the verification circuit of the composed SNARK is smaller than the so-called *recursion threshold* of the base SNARK \mathcal{I} . This refers to the smallest circuit size S^* such that the verification procedure of \mathcal{I} cannot be represented by a circuit-satisfiability instance of size smaller than S^* . On circuits smaller than the recursion threshold, composing the SNARK with itself does not reduce verification costs (and in fact may increase them).

Of course, the deeper the recursion, the more work the prover has to do. For example, if \mathcal{I} is composed with itself three times, then the prover has to “in its own head” first produce a proof π that would convince the \mathcal{I} verifier of the claim at hand, then produce a proof π' that it knows π , then produce a proof π'' that it knows π' .¹⁵³ This is naturally more work than just producing the proof π for the non-composed proof system.

Establishing knowledge soundness of composed SNARKs. When considering the composition \mathcal{F} of two SNARKs \mathcal{I} and \mathcal{O} (Section 15.1), we presented \mathcal{F} in a manner that hopefully made intuitively clear that it is knowledge-sound: the \mathcal{F} -prover $\mathcal{P}_{\mathcal{F}}$ establishes using the outer SNARK \mathcal{O} that it knows a proof π that would have caused the \mathcal{I} -verifier to accept the claim at hand, namely that $\mathcal{P}_{\mathcal{F}}$ knows a w such that $\mathcal{C}(w) = 1$. In turn, since \mathcal{I} is knowledge-sound, any efficient party $\mathcal{P}_{\mathcal{F}}$ who knows such a proof π must also know such a witness w .

Still, it is instructive to carefully write out a description of the procedure $\mathcal{E}_{\mathcal{F}}$ that extracts the witness w from $\mathcal{P}_{\mathcal{F}}$. This will help us understand knowledge-extraction for the “deeper” compositions considered in this section. As we will see, the natural knowledge extractor for a composed SNARK will have runtime that grows exponentially with the depth of the composition. This means that super-constant depth compositions

¹⁵³By in its own head, we mean the prover performs a computation without sending the result to the verifier.

will yield a superpolynomial-time knowledge-extractor. Hence, the knowledge-soundness of such deep compositions is not on firm theoretical footing.¹⁵⁴

Knowledge extractor for $\mathcal{F} = \mathcal{O} \circ \mathcal{I}$. Given an efficient prover $\mathcal{P}_{\mathcal{F}}$ that can generate accepting proofs for \mathcal{F} , $\mathcal{E}_{\mathcal{F}}$ must identify a witness w such that $\mathcal{C}(w) = 1$. $\mathcal{E}_{\mathcal{F}}$ works as follows. Since a convincing proof for \mathcal{F} establishes via the outer SNARK system \mathcal{O} that \mathcal{P} knows a proof π causing the \mathcal{I} -verifier to accept, $\mathcal{E}_{\mathcal{F}}$ can first apply the following sub-routine: “run the knowledge-extractor $\mathcal{E}_{\mathcal{O}}$ for \mathcal{O} to extract from $\mathcal{P}_{\mathcal{F}}$ such a proof π ”. This sub-routine itself represents an efficient convincing prover algorithm $\mathcal{P}_{\mathcal{I}}$ for the inner SNARK \mathcal{I} . Hence, $\mathcal{E}_{\mathcal{F}}$ can apply the knowledge-extractor $\mathcal{E}_{\mathcal{I}}$ to extract from $\mathcal{P}_{\mathcal{I}}$ a witness w such that $\mathcal{C}(w) = 1$.

How efficient is $\mathcal{E}_{\mathcal{F}}$? $\mathcal{E}_{\mathcal{F}}$ has to apply the inner-SNARK knowledge-extractor $\mathcal{E}_{\mathcal{I}}$ to a prover $\mathcal{P}_{\mathcal{I}}$ that itself runs the outer-SNARK knowledge-extractor $\mathcal{E}_{\mathcal{O}}$ on $\mathcal{P}_{\mathcal{F}}$. Hence, $\mathcal{E}_{\mathcal{F}}$ may be significantly slower than $\mathcal{E}_{\mathcal{I}}$ or $\mathcal{E}_{\mathcal{O}}$ individually (though $\mathcal{E}_{\mathcal{F}}$ still runs in polynomial time as long as $\mathcal{E}_{\mathcal{I}}$ and $\mathcal{E}_{\mathcal{O}}$ both do). For example, if A denotes the number of times $\mathcal{E}_{\mathcal{I}}$ calls¹⁵⁵ $\mathcal{P}_{\mathcal{I}}$ to extract w from it, and B denotes the number of times $\mathcal{E}_{\mathcal{O}}$ calls $\mathcal{P}_{\mathcal{F}}$ to extract π from it, then the entire extraction procedure $\mathcal{E}_{\mathcal{F}}$ may call $\mathcal{P}_{\mathcal{F}}$ up to $A \cdot B$ times.¹⁵⁶

Knowledge extractor for deeper compositions. Now consider a SNARK \mathcal{O} composed with itself, say, four times, and denote the composition by $\mathcal{O}^4 := \mathcal{O} \circ \mathcal{O} \circ \mathcal{O} \circ \mathcal{O}$. We can view \mathcal{O}^4 as $\mathcal{O}^2 \circ \mathcal{O}^2$, where $\mathcal{O}^2 := \mathcal{O} \circ \mathcal{O}$. The previous paragraph shows that if A denotes the number of times that the knowledge extractor $\mathcal{E}_{\mathcal{O}}$ for \mathcal{O} must run a convincing prover $\mathcal{P}_{\mathcal{O}}$ to extract a witness, then the number of times that the natural knowledge extractor for \mathcal{O}^2 must run a convincing prover $\mathcal{P}_{\mathcal{O}^2}$ to extract a witness is A^2 . Then applying the same analysis to $\mathcal{O}^2 \circ \mathcal{O}^2$ means that the number of times the natural knowledge extractor for \mathcal{O}^4 must run a convincing prover is A^4 .

In general, composing \mathcal{O} with itself t times will yield a knowledge extractor that runs a prover generating convincing proofs at most A^t times. If A is polynomial in the size of the statement that the SNARK is applied to, then A^t will be superpolynomial unless t is constant.¹⁵⁷

Practical considerations of composition. For many popular SNARKs \mathcal{O} , there can be considerable concrete overhead in attempting to represent the \mathcal{O} -verifier as an equivalent instance of arithmetic circuit-satisfiability or R1CS, or whatever intermediate representation is “consumed” by the outer SNARK. Here, we highlight one particularly common and important issue, and describe how it has been addressed to date.

As we have seen in Chapters 13 and 14, many popular SNARKs require the verifier to perform operations in cryptographic groups in which the discrete logarithm problem is intractable (and for many SNARKs, the groups must furthermore be pairing-friendly, see Section 13.2.1). Modern instantiations of such cryptographic groups use elliptic curves (Section 11.1.2). Recall that elements of an elliptic curve group corre-

¹⁵⁴While we cannot prove knowledge-soundness of superconstant-depth SNARK recursions, that does not necessarily mean we think deep recursions are *not* knowledge-sound, just that we don’t know how to provably reduce their knowledge-soundness to that of the underlying base SNARK. Indeed deep recursions of SNARKs are beginning to see practical deployment in distributed environments (e.g., [BMRS20b]). See also Sections 15.4 and 15.5).

¹⁵⁵When we say that a knowledge extractor “calls” a prover more than once, we refer to the fact that the extractor might repeatedly “rewind and restart” the prover from which it is extracting a witness. We saw examples of this in the context of interactive protocols in Remark 11.1 in Section 11.2.1 and in Section 13.1.4.1, and extractors for SNARKs obtained by applying the Fiat-Shamir transformation to an interactive protocol also rewind and restart the prover in a similar manner (Section 11.2.3). SNARKs that rely on knowledge-of-exponent assumptions rather than the Fiat-Shamir transformation (e.g., Sections 13.2.2 and 14.5.4), also rely on extractors that may run the prover multiple times before extracting a witness (such an extractor is essentially posited by the knowledge-of-exponent assumption itself).

¹⁵⁶ A and B may depend on the size of the statement being proven, but we suppress this dependence from our notation for simplicity.

¹⁵⁷If A is a constant independent of the statement size, then t can be logarithmic while keeping A^t polynomial.

spond to pairs of points $(x, y) \in \mathbb{F} \times \mathbb{F}$ that satisfy an equation of the form $y^2 = x^3 + ax + b$ for field elements a and b . \mathbb{F} is referred to as the *base field* of the curve. When designing a discrete-logarithm-based SNARK for arithmetic circuit-satisfiability or RICS-satisfiability over a field \mathbb{F}_p of prime order p , one requires that the *order* of the elliptic curve group \mathbb{G} be p (in this case, \mathbb{F}_p is called the *scalar field* of \mathbb{G}). The crucial point here is that the base field \mathbb{F} and the scalar field \mathbb{F}_p of \mathbb{G} are *not* the same field (see Footnote 114). This means that, in a discrete-log-based SNARK \mathcal{O} for an arithmetic circuit \mathcal{C} defined over field \mathbb{F}_p , the verifier has to perform field operations over a base field \mathbb{F} that *differs* from \mathbb{F}_p .

Recall that in order to compose a SNARK \mathcal{O} for circuit-satisfiability with itself, one must represent the verification procedure of \mathcal{O} as an arithmetic circuit \mathcal{C}' to which \mathcal{O} can be applied. If \mathcal{O} uses a cryptographic group \mathbb{G} as per the above paragraph, then it is natural to define \mathcal{C}' over the base field \mathbb{F} of \mathbb{G} rather than the scalar field \mathbb{F}_p of \mathbb{G} , so that \mathcal{C}' can “natively” perform the operations over \mathbb{F} required to perform group operations in \mathbb{G} (while it is possible to “implement” \mathbb{F} operations via a circuit defined over a different field \mathbb{F}_p using techniques discussed in Chapter 5, it is currently quite expensive, despite efforts from many researchers to make it less so). But in order to apply \mathcal{O} to \mathcal{C}' , one needs to know *another* cryptographic group \mathbb{G}' whose scalar field (rather than base field) is \mathbb{F} .

Accordingly, to support arbitrary-depth compositions of \mathcal{O} with itself (or with other SNARKs), it is useful to identify a *cycle* of elliptic curves. The simplest form of such a cycle has length two. This is a pair of elliptic curve groups \mathbb{G} and \mathbb{G}' such that the base field \mathbb{F}_p of \mathbb{G} is the scalar field \mathbb{F} of \mathbb{G}' and vice versa. Using such a cycle of elliptic curves ensures that the verifier of \mathcal{O} applied to a circuit over field \mathbb{F} can be efficiently implemented via a circuit over field \mathbb{F}_p , and vice versa.

To walk through the specific example of depth-two recursive composition: let \mathcal{O} be a SNARK for arithmetic circuit-satisfiability. It will be helpful to use a subscript $\mathcal{O}_{\mathbb{F}}$ to clarify what field the circuit-satisfiability instance is defined over. Then $\mathcal{O}^3 := \mathcal{O}_{\mathbb{F}_p} \circ \mathcal{O}_{\mathbb{F}} \circ \mathcal{O}_{\mathbb{F}_p}$ will work as follows to establish knowledge of a w such that $\mathcal{C}(w) = 1$, where \mathcal{C} is defined over \mathbb{F}_p . First, the \mathcal{O}^3 prover \mathcal{P} in its own head will generate a proof π that convinces the $\mathcal{O}_{\mathbb{F}_p}$ -verifier of the claim. The $\mathcal{O}_{\mathbb{F}_p}$ verifier for this claim can be efficiently represented by a circuit \mathcal{C}' over \mathbb{F} . So (in its own head once again) the \mathcal{O}^3 prover will generate an $\mathcal{O}_{\mathbb{F}}$ -proof π' that it knows such an $\mathcal{O}_{\mathbb{F}_p}$ -proof π . The $\mathcal{O}_{\mathbb{F}}$ -verifier for this claim can in turn be efficiently implemented by a circuit \mathcal{C}'' over \mathbb{F}_p , so the \mathcal{O}^3 prover finally computes a proof π'' that it knows such an $\mathcal{O}_{\mathbb{F}}$ -proof π' . And \mathcal{P} sends this proof explicitly to the \mathcal{O}^3 verifier.

More generally, given a cycle of elliptic curves, arbitrary-depth composition of $\mathcal{O}_{\mathbb{F}}$ and $\mathcal{O}_{\mathbb{F}_p}$ can be supported. Every time the prover needs to produce a proof π' that it knows a proof π that the $\mathcal{O}_{\mathbb{F}_p}$ -verifier would accept, it represents the $\mathcal{O}_{\mathbb{F}_p}$ -verifier as a circuit over \mathbb{F} and applies the $\mathcal{O}_{\mathbb{F}}$ SNARK to this circuit, and similarly with the roles of \mathbb{F}_p and \mathbb{F} reversed.

Currently, a popular cycle of (non-pairing-friendly) curves are Pasta curves¹⁵⁸, which are reasonably close in efficiency to some of the best curves that don’t support cycles (e.g., Curve25519, see Section 11.1.2). Cycles of pairing-friendly curves are also known, e.g., via so-called MNT curves [CCDW20], but, at the time of writing, for a given security level these remain significantly less efficient than popular pairing-friendly curves for SNARK design that don’t support cycles (e.g., BLS12-381, see Section 13.2.1). This owes to a need of the cycle-supporting curves to work over significantly larger finite fields, which leads to slower group operations.

Another common practical consideration arising in recursive SNARK composition is that the verifier in many transparent SNARKs performs Merkle hash path verifications, which means cryptographic hash operations must be expressed as a circuit- or RICS-satisfiability instance. As mentioned in Chapter 5, there has been considerable effort devoted to developing “SNARK-friendly” hash functions, meaning plausibly

¹⁵⁸<https://electriccoin.co/blog/the-pasta-curves-for-halo-2-and-beyond/>

collision-resistant hash functions that can be efficiently expressed in such a form.

15.3 Other Applications of SNARK Composition

We have seen that composition of SNARKs can be used to improve efficiency: a SNARK with fast prover and somewhat slow verification can be composed with itself or with another SNARK to improve the verification costs. There are other reasons to compose SNARKs.

Incremental computation. One, which we will in detail later in this chapter (Sections 15.4 and 15.5) use recursion more directly to construct efficient SNARKs tailored for iterative computation, i.e., to prove that for some designated input x and specified function F that $F(F(F(F(F(F(x)))))) = y$. More generally, let $F^{(i)}(x)$ denote the i -fold iterative application of F to x , e.g., $F^{(3)}(x) = F(F(F(x)))$. A quintessential application of such proof systems is to let F be a *delay function*, meaning a simple function that requires some non-trivial sequential computation to compute. Then a SNARK for many iterative applications of F yields a *verifiable delay function*: a function that requires significant sequential time to compute, yet the result of which can be efficiently verified.

Incrementally Verifiable Computation (IVC). Certain applications (to be discussed momentarily) actually call for a primitive called *incrementally verifiable computation* [Val08]. This means that after each application j of F to x , a prover can output y_j and a SNARK proof π_j that $F^{(j)}(x) = y_j$, and moreover, given y_j and π_j , any other party can apply F to y_j to obtain an output y_{j+1} and efficiently compute a new SNARK proof π_{j+1} that $F^{(j+1)}(x) = y_{j+1}$.

Applications to distributed computing environments. In fact, our SNARKs for iterative computation will be able more generally to handle non-deterministic computations F . That is, F can take two inputs, a public input x and a witness w , and produce some output $y = F(x, w)$. The SNARKs we present hereon in this chapter will be able to¹⁵⁹ establish knowledge of witnesses w_1, \dots, w_i such that

$$F(F(\dots F(F(F(x, w_1), w_2), w_3), \dots, w_{i-1}), w_i) = y_i.$$

Here is one example of a possible application to public blockchains. Think of F as taking as input the current state of an “accumulation” (e.g., Merkle-hash, see Section 6.3.2.2) of all account balances for a public blockchain, and think of each witness w_i as specifying a new valid transaction t_i along with associated proof-of-work, and such that F outputs an updated accumulation (i.e., outputs the accumulation of the new account balances following the processing of transaction t_i). Then a SNARK for the above yields a proof that y_i is a valid accumulation of account balances after i transactions. This can enable computationally weak nodes in a blockchain network to very efficiently learn (from any untrusted party) an accumulation of the global state of the network (i.e., the current account balances), with a proof that the accumulation actually captures a sequence consisting of a certain number of valid transactions and associated proofs-of-work (this may be important for protocols that designate the current state of the network to be that of the “longest chain”, i.e., the longest known sequence of valid transactions). Hence, nodes can trustlessly learn the accumulation of the network state, with no need to download the entire transaction history of the network or even the current account balances.

¹⁵⁹For simplicity, we do not present the SNARKs in this level of generality but they will support it without modification.

Proof aggregation. Another application of SNARK composition is proof aggregation, which can be explained via the following example application. Suppose that a prover \mathcal{P} claims for some public input x and function F that $F(x) = y$, but computing F is highly computation-intensive. Imagine that the computation is broken up into ℓ more manageable pieces, say, $F_1(x), \dots, F_\ell(x)$, that can be performed independently of each other. \mathcal{P} farms each piece out to a different machine (possibly untrusted even by the prover, who is in turn untrusted by the verifier), to produce outputs y_1, \dots, y_ℓ , which are then combined via some aggregation function G to produce the final output y .

In order to prove that $F(x) = y$, each machine can produce a proof π_i that $y_i = F_i(x)$, and send both π_i and the result y_i back to \mathcal{P} ; then it suffices for \mathcal{P} to (a) prove knowledge of the convincing proofs π_1, \dots, π_ℓ for the ℓ claims $y_i = F_i(x)$, and (b) prove that $G(y_1, \dots, y_\ell) = y$. One can accomplish this by applying a SNARK to the computation that first *verifies* the proofs π_1, \dots, π_ℓ and then computes $G(y_1, \dots, y_\ell)$.

15.4 SNARKs for Iterative Computation via Recursion

Recall that $F^{(i)}(x)$ denote the i -fold iterative application of F to x . Suppose we want to design a SNARK for the claim that $F^{(i)}(x) = y$.

One could of course apply any of the (non-composed) SNARKs from earlier chapters of this survey to $F^{(i)}$, but these come with various downsides and tradeoffs (delineated in detail in the next chapter, Chapter 16). For starters, they do not support IVC (Section 15.3). Turning to efficiency, if one desires the shortest possible proofs and fastest verification, the SNARKs with these properties require a trusted setup (see Chapter 14). They also tend to be quite space-intensive for the prover due in part to their use of FFTs, so applying them to very large computations may not be feasible, and their use of pairing-friendly groups can lead to slow prover time. While many of the transparent SNARKs of earlier chapters avoid FFTs and pairings, they have much larger proofs and verification costs than the trusted-setup SNARKs with fastest verification.

The recursive-composition-of-SNARKs approach. Can we address the above issues by taking a base SNARK \mathcal{O} and applying recursive composition? Let us imagine for a moment that we have already designed a SNARK \mathcal{O}_{i-1} for the claim that $F^{(i-1)}(x) = y_{i-1}$. Then here is a SNARK \mathcal{O}_i for the claim that $F^{(i)}(x) = y_i$: the prover \mathcal{P} uses the base SNARK \mathcal{O} to prove that

- (a) it knows an \mathcal{O}_{i-1} -proof π_{i-1} that $F^{(i-1)}(x) = y_{i-1}$, and
- (b) that $F(y_{i-1}) = y_i$.¹⁶⁰

¹⁶⁰An important practical issue here is that, in order to identify a single arithmetic circuit confirming both (a) and (b), it is essential that the \mathcal{O}_{i-1} -verifier's computation and F itself both be efficiently expressible as a circuit over the same field. This can be challenging for SNARKs that perform elliptic curve operations, because as discussed in Section 15.2, such SNARK verifiers are only efficiently representable as circuits over the *base* field \mathbb{F} of the curve, which differs from the (scalar) field \mathbb{F}_p that F is presumably efficiently representable over. One way to sidestep this issue is to identify a cycle of curves with scalar and base fields \mathbb{F}_p and \mathbb{F} such that F is efficiently computable by circuits over *both* \mathbb{F}_p and \mathbb{F} . This way, at each step i , the \mathcal{O}_{i-1} verifier will be efficiently expressible as a circuit over one of the two fields (which one depends on whether i is odd or even), and F will also be efficiently expressible as a circuit over the same field. If F is only efficiently computable by a circuit over \mathbb{F}_p , then one will run into the issue that the \mathcal{O}_{i-1} -verifier is efficiently representable as a circuit only over \mathbb{F} , and F itself is not. To address this, one can define \mathcal{O}_i via *two* steps of SNARK composition, rather than one. In the first step, the prover represents the \mathcal{O}_{i-1} -verifier as a circuit over \mathbb{F} , and in its own head computes an $\mathcal{O}_{\mathbb{F}}$ -proof π that it knows an \mathcal{O}_{i-1} -proof π_{i-1} that $F^{(i-1)}(x) = y_{i-1}$. Then, since (unlike the \mathcal{O}_{i-1} -verifier) the $\mathcal{O}_{\mathbb{F}}$ -verifier is efficiently representable as a circuit over \mathbb{F}_p , there is a small circuit over \mathbb{F}_p to establish that both (a) the prover knows such a proof π and (b) that $F(y_{i-1}) = y_i$. Hence, $\mathcal{O}_{\mathbb{F}_p}$ can be applied to this circuit to yield a proof that $F^{(i)}(x) = y_i$.

This recursive-composition-of-SNARKs approach to incremental computation has been pursued (e.g., [BCTV14a]) using the trusted-setup SNARK with fastest known verification, which is now due to Groth [Gro16] (Section 14.5.5).¹⁶¹ A major benefit of the recursive approach is that it yields IVC (Section 15.3): for each iteration $j-1$, the prover could output y_{j-1} and the proof π_{j-1} that $y_{j-1} = F^{(j-1)}(x)$, and any other party could “pick up the computation from there”, computing $F(y_{j-1})$ and using π_{j-1} to compute a proof π_j that $y_j = F^{(j)}(x)$.

Relative to the direct application of the non-composed base SNARK, the above recursive solution also reduces the prover’s space cost, because the prover only ever applies the base SNARK to a *single* application of F , one after the other (i.e, it does not apply the base SNARK “all at once” to an entire circuit computing $F^{(i)}$). That is, at any time j during its computation of the proof π_i , the prover only needs to remember the preceding proof π_{j-1} and the preceding output y_{j-1} of $F^{(j-1)}$.

On the other hand, a significant downside of the recursive approach when applied to a SNARK that uses pairings such as Groth’s [Gro16] is that the prover is quite slow, in large part owing to the need to use cycles of pairing-friendly elliptic curves to support arbitrary-depth recursion (Section 15.2). On top of this, there is additional overhead for the prover that can be traced to a notion we term the *overhead of recursion*.

The overhead of recursion. Effectively, the final SNARK proof π_i for $F^{(i)}$ establishes that for all $j \leq i$, the prover \mathcal{P} not only faithfully applied F to y_{j-1} to obtain y_j (as per (b) above), but *also* that \mathcal{P} , in its own head, faithfully verified the proof π_{j-1} as per (a) above.¹⁶² Put another way, the above recursive approach replaces the computation of $F(y_{j-1})$ with a *larger* computation $F'(y_{j-1}, \pi_{j-1})$ that outputs $F(y_{j-1})$ and verifies π_{j-1} , and it applies the base SNARK to F' for all $j \leq i$. (This perspective will come up again in Section 15.5).

We refer to the added cost to the prover of establishing that it verified π_{j-1} for each iterative application j of F as the “overhead of recursion”. This is because non-recursive solutions (i.e., a direct application of a SNARK to a circuit computing $F^{(i)}$) require the prover to establish only that it faithfully applied F all i times, not that it verified any proofs of its own faithfulness along the way. Hence, the “overhead of recursion” is purely extra work for the prover, which does not arise in non-recursive solutions.

This overhead is naturally measured by the number of gates in a circuit (or other intermediate representation as appropriate) implementing the base SNARK’s verifier.¹⁶³ This will be the dominant contributor to the prover’s costs if this circuit is larger than the circuit required to implement F itself. Specifically, this happens if the circuit representing F is smaller than the *recursion threshold* of the base SNARK \mathcal{O} (see Section 15.2).

Trusted-setup SNARKs with state-of-the-art verification costs [Gro16] have a reasonably low recursion threshold. Still, we will see later (Section 15.5) that this overhead can be reduced further via other approaches that moreover can avoid a trusted setup and the use of pairings (the use of which both increases the recursion threshold and, as mentioned above, leads to concretely high prover costs).¹⁶⁴

¹⁶¹More recent work has studied recursive-composed SNARKs with a universal rather than circuit-specific trusted setup, but this leads to even higher overheads for the prover [CCDW20].

¹⁶²To clarify, π_i establishes all of this without even “telling the verifier” what y_{j-1} or π_{j-1} even were.

¹⁶³The issue described at the end of Footnote 160 can further increase the overhead of recursion, by forcing *two* statements about SNARK verification circuits to be proved for every application of F , rather than one.

¹⁶⁴Verification of Groth’s SNARK [Gro16] involves 3 pairing computations, which are concretely fairly expensive, especially once represented as a circuit or R1CS. Hence, there is room to reduce this overhead further. We will see an approach later in this chapter (Section 15.5) that reduces the “three pairing computations” down to roughly two group exponentiations (in a *non*-pairing-friendly group), which concretely can be represented by a significantly smaller circuit or R1CS than three pairing computations.

Recursively composing transparent SNARKs. To recap, there are a number of downsides to above approach of recursively composing a SNARK with state-of-the-art verification costs: the base SNARK’s need for a trusted setup, the very high prover overheads due to the use of cycles of pairing-friendly curves, and the concretely sub-optimal “overhead of recursion”.

The most straightforward approach to address the first two issues is to replace the trusted-setup SNARKs with transparent SNARKs that moreover do not require pairing-friendly groups. These SNARKs all utilize transparent polynomial commitment schemes (e.g., based on FRI (Section 9.4), Ligeró’s polynomial commitment scheme (Section 9.6), Hyrax’s polynomial commitment scheme (Section 13.1.3), or Bulletproofs (Section 13.1.4)). The problem with a naive implementation of this approach is that the verification of evaluation proofs of such polynomial commitment schemes is quite expensive and hence the overhead of recursion is very large. For example, if the popular Bulletproofs polynomial commitment is used, then while proofs are short (logarithmic in size), the verification cost is linear. Even FRI-based polynomial commitments (Section 9.4), while achieving polylogarithmic proof size, are concretely quite large for appropriate security levels, and involve many Merkle hash path authentication operations, which can be somewhat expensive to represent as a circuit or R1CS (see the end of Section 15.2).

To address the overhead of recursion in this case, a line of works starting with Halo [BGH19, BDFG21, BCL⁺21, BCMS20, KST21] has roughly shown how to avoid feeding verification of evaluation proofs of polynomial commitment schemes through the proof machinery. The verifier in these transparent SNARKs can be split into two parts: (a1) verifying all parts of the proof other than evaluations of committed polynomials and (a2) verifying evaluations of committed polynomials. Essentially, the SNARK is modified to simply *omit* the verification check (a2). This means that, each time the prover, in its own head, generates a “proof”¹⁶⁵ π_j that $F^{(j)}(x) = y_j$ (having already computed a “proof” π_{j-1} that $F^{(j-1)}(x) = y_{j-1}$), π_j does *not* directly attest to the validity of any claimed evaluations of committed polynomials involved in the “proof”. So these evaluation claims must be checked separately. What these works roughly do is show how to use homomorphism properties of known polynomial commitment schemes to cheaply “batch-check” *all* evaluations of *all* committed polynomials across all “proofs” π_1, \dots, π_i that the prover generated in its own head. That is, all such evaluation claims regarding committed polynomials across π_1, \dots, π_i are “accumulated” into a single claim, which can then be checked at the same cost as a *single* claim.

The most recent works in this line have taken the above approach to its logical extreme and derived SNARKs for iterative computation $F^{(i)}$ purely from homomorphic vector commitment schemes (i.e., without first developing a “base SNARK” that is recursively applied i times). See Footnote 175 in Section 15.5.4 for additional discussion of this perspective. The following section describes one such result, yielding a proof system called Nova [KST21].

15.5 SNARKs for Iterative Computation via Homomorphic Commitments

Our goal in this section is to design a SNARK for iterative computation directly from homomorphic vector commitment schemes. The resulting SNARK is transparent, avoids the need for pairing friendly curves, and has state-of-the-art overhead of recursion. These last two properties together ensure a significantly faster prover relative to the recursive composition of pairing-based SNARKs (Section 15.4).

¹⁶⁵Here, we are putting the word “proof” in quotes, because π_j omits essential verification information, namely verification of evaluations of committed polynomials; hence, π_j is not actually a complete SNARK proof for the claim at hand, that $F^{(j)}(x) = y_j$.

15.5.1 Informal Overview of the SNARK

The SNARK will roughly work as follows. Using the front-end techniques of Chapter 5, one first transforms F into an equivalent RICS instance, i.e., three public matrices $A, B, C \in \mathbb{F}^{n \times n}$ such that $F(x) = y$ if and only if there exists a vector z of the form (x, y, w) for some witness w such that $(A \cdot z) \circ (B \cdot z) = C \cdot z$.¹⁶⁶

Let $y_0 = x$. Then proving that $F^{(i)}(x) = y_i$ is equivalent to showing the existence of vectors w_1, \dots, w_i such that for

$$z_j := (y_{j-1}, y_j, w_j), \quad (15.1)$$

$$(A \cdot z_j) \circ (B \cdot z_j) = C \cdot z_j: j = 1, \dots, i. \quad (15.2)$$

The rough idea of the SNARK is that \mathcal{P} will commit to all of the vectors z_1, \dots, z_i (using a homomorphic vector-commitment scheme), and prove that each one has the form Equation (15.1) and satisfies Equation (15.2). It will do this by repeatedly applying a primitive called a “folding scheme”—roughly, a way of taking two RICS instances of the form Equation (15.2) and transforming them into a single RICS instance such that the derived instance is satisfied if and only if both original instances are satisfied.¹⁶⁷ The folding scheme can be repeatedly applied to reduce all i instances of Equation (15.2) into a single instance. For simplicity, we will focus on the “sequential” folding pattern whereby instance one of Equation (15.2) is folded with instance two, and then the resulting derived instance is folded with instance three, and then the resulting derived instance is folded with instance four, and so on until all i instances have been folded into a single one.¹⁶⁸ The folding scheme is interactive, but the interaction can be removed with the Fiat-Shamir transformation.

The validity of this final RICS instance can be proven with any SNARK for RICS instances of the form $(A \cdot z) \circ (B \cdot z) = C \cdot z$ in which the prover commits to the witness vector z via the same homomorphic vector commitment scheme used by the prover to commit to z_1, \dots, z_i . This includes, for example, SNARKs that make use of the Bulletproofs polynomial commitment scheme (Section 13.1.4) as the commitment in Bulletproofs is just a generalized Pedersen commitment to the coefficient vector of the polynomial. If Bulletproofs is used, the length of the SNARK proof for the final RICS instance that results from folding can be made $O(\log n)$, though the verification time will be $O(n)$.¹⁶⁹

The above brief description glosses over a number of details. First, the folding scheme will take two RICS instances and not yield another RICS instance, but rather a generalization that we call *committed-RICS-with-a-slack-vector*. Second, because each folding operation will require a message from the prover

¹⁶⁶Previous sections in this chapter referred to SNARKs for arithmetic circuit satisfiability for simplicity and concreteness, but as pointed out in Footnote 152, they apply without modification to SNARKs for RICS. In this section, we use the formalism of RICS rather than circuits because Nova is most naturally described in the RICS setting. Of course, RICS is a generalization of a circuit (see Section 7.5), so any SNARK for RICS representations also yields a SNARK for circuit representations.

¹⁶⁷This folding scheme is reminiscent several earlier protocols in this text. Specifically, in each round of Bulletproofs (Section 13.1.4), a claim about an inner product of committed vectors of length n is reduced to a derived claim about an inner product of vectors of length $n/2$. Also, in each round of the sum-check protocol (Section 4.2), a claim about a sum over 2^ℓ terms is reduced to a claim about a sum over $2^{\ell-1}$ terms. In fact, there have been works that view these protocols through a unified lens [BCS21, KP22].

¹⁶⁸In general, any folding pattern can be used. That is, we can treat the i instances as the leaves of any binary tree, with any internal node of the tree representing the “folding” of its two children into a single instance. The root of the tree represents the final RICS instance that results from all of the folding operations.

¹⁶⁹For iterative computation, one typically thinks of the number of iterations i as very large, and function F applied at each iteration as small, perhaps even computed by a constant-size circuit. In this case, $O(n)$ can be thought of as a constant and $O(\log n)$ as an *even smaller* constant.

to the verifier (and a random challenge sent from verifier to prover), the proof length of the resulting protocol will be linear in i , when we would really like a proof length that is *independent* of i . We will ultimately achieve the desired proof length via a variant of recursive proof composition (Section 15.5.4). We additionally have not explained how to check that each committed vector z_j has the form of Equation (15.1).

15.5.2 A Folding Scheme for Committed-R1CS-with-a-Slack-Vector

The problem of committed-R1CS-with-a-slack-vector. In an instance of this problem, there are three public $n \times n$ matrices A, B , and C with entries from a field \mathbb{F} , as well as a public scalar $u \in \mathbb{F}$ and a public vector $s \in \mathbb{F}^m$. In addition to those public objects, there are two committed vectors $w \in \mathbb{F}^{n-m}$ and E in \mathbb{F}^n . Let $z = (s, w) \in \mathbb{F}^n$. One should think of the prover as having already committed to w and E using a homomorphic vector-commitment scheme (e.g., Pedersen vector commitments from Section 13.1.2). The prover claims that

$$(A \cdot z) \circ (B \cdot z) = u \cdot (C \cdot z) + E.$$

Consider having two instances of committed-R1CS-with-a-slack-vector, in which the public matrices in the two instances are identical. That is, the prover has claimed that:

$$(A \cdot z_1) \circ (B \cdot z_1) = u_1 \cdot C \cdot z_1 + E_1, \tag{15.3}$$

$$(A \cdot z_2) \circ (B \cdot z_2) = u_2 \cdot C \cdot z_2 + E_2. \tag{15.4}$$

Here, $A, B, C \in \mathbb{F}^{n \times n}$ are public matrices and $u_1, u_2 \in \mathbb{F}$ are public scalars, $s_1, s_2 \in \mathbb{F}^m$ are public vectors, $w_1, w_2 \in \mathbb{F}^{n-m}$ and $E_1, E_2 \in \mathbb{F}^n$ are committed vectors, and $z_1 = (s_1, w_1)$ and $z_2 = (s_2, w_2)$. \mathcal{V} would like to check both of these claims. The naive way to do this would be to have the prover open the commitments to w_1, w_2, E_1 , and E_2 , so \mathcal{V} can check both claims directly, but this naive approach is too expensive for our purposes. Instead, imagine the verifier \mathcal{V} would like to “take a random linear combination” of the two claims, to derive a single claim of the same form, such that the derived claim is true (up to some negligible soundness error) if and only if both of the original claims are true.

Here is a way the verifier could try to accomplish this.

A first attempt that doesn’t work. The verifier could choose a random field element $r \in \mathbb{F}$, and let

$$s \leftarrow s_1 + r \cdot s_2 \tag{15.5}$$

$$w \leftarrow w_1 + r \cdot w_2 \tag{15.6}$$

$$u \leftarrow u_1 + r \cdot u_2 \tag{15.7}$$

$$E \leftarrow E_1 + r^2 E_2. \tag{15.8}$$

Observe that \mathcal{V} can directly compute s and u because $s_1, s_2 \in \mathbb{F}^m$ and $u_1, u_2 \in \mathbb{F}$ are public. Also, by homomorphism of the commitment scheme used by \mathcal{P} to commit to w_1, w_2, E_1 , and E_2 , the verifier can on its own compute commitments to w and E . The verifier might *hope* that under these definitions, Equation (15.3) and (15.4) imply the following (and vice versa):

$$(A \cdot z) \circ (B \cdot z) = u \cdot (C \cdot z) + E. \quad (15.9)$$

If this were the case, then the verifier, on its own, could derive a single new instance of committed-RICS-with-a-slack-vector that is *equivalent* to the validity of the two original instances (Equations (15.3) and (15.4)).

Unfortunately, even if Equation (15.3) and (15.4) both hold, Equation (15.9) does *not* hold. But as we will see, we can slightly modify the definition of E so that Equation (15.9) does hold.

What does work. Let us redefine E to include an extra “cross-term”, namely, throw away Equation (15.8) and replace it with:

$$E \leftarrow E_1 + r^2 E_2 + r \cdot T \quad (15.10)$$

where

$$T \leftarrow (Az_2) \circ (B \cdot z_1) + (Az_1) \circ (B \cdot z_2) - u_1 \cdot C \cdot z_2 - u_2 \cdot C \cdot z_1. \quad (15.11)$$

Then it can be checked via elementary algebra that Equation (15.9) holds for every choice of $r \in \mathbb{F}$.

Calculation showing that Equation (15.9) holds for every $r \in \mathbb{F}$. The left hand side of Equation (15.9) is:

$$\begin{aligned} (A \cdot z) \circ (B \cdot z) &= (A \cdot z_1 + r \cdot A \cdot z_2) \circ (B \cdot z_1 + r \cdot B \cdot z_2) \\ &= (A \cdot z_1) \circ (B \cdot z_1) + r^2 \cdot (A \cdot z_2) \circ (B \cdot z_2) + r \cdot ((A \cdot z_1) \circ (B \cdot z_2) + (A \cdot z_2) \circ (B \cdot z_1)) \end{aligned} \quad (15.12)$$

while the right hand side equals:

$$u \cdot (C \cdot z) = (u_1 + r u_2) \cdot C \cdot (z_1 + r z_2) = u_1 \cdot C \cdot z_1 + E_1 + r^2 (u_2 \cdot C \cdot z_2 + E_2) + r (u_2 \cdot C \cdot z_1 + u_1 \cdot C \cdot z_2) \quad (15.13)$$

By Equations (15.3) and (15.4), we can rewrite Expression (15.12) as:

$$u_1 \cdot C \cdot z_1 + E_1 + r^2 \cdot (u_2 \cdot C \cdot z_2 + E_2) + r \cdot ((A \cdot z_1) \circ (B \cdot z_2) + (A \cdot z_2) \circ (B \cdot z_1)). \quad (15.14)$$

The difference between Expression (15.14) and the right hand side of Equation (15.13) is exactly r times the value assigned to T by Equation (15.11).

Accordingly, consider the following simple interactive protocol that seeks to “reduce” checking that Equation (15.3) and (15.4) both hold to the task of checking that Equation (15.9) holds: First, \mathcal{P} commits to a vector v claimed to equal the cross-term T (Equation (15.11)) using the same homomorphic vector-commitment scheme used to commit to w_1, w_2, E_1 , and E_2 . Next, \mathcal{V} chooses r at random from \mathbb{F} and sends it to \mathcal{P} . Observe that, given the commitments to E_1, E_2 , and v , \mathcal{V} can use the homomorphism to compute a commitment to the vector $E_1 + r^2 E_2 + r \cdot v$ (which, if v is as claimed, equals the right hand side of Equation (15.10)).

We have already explained that if the committed vector v equals T (Equation (15.11)) as prescribed, then Equation (15.10) holds with probability 1 over the random choice of r . Meanwhile, it is not hard to see that if the prover commits to a vector v that *differs* from T , then with probability $1 - 2/|\mathbb{F}|$ over the random choice of r , Equation (15.9) will fail to hold. This is because, if $v_j \neq T_j$ for some $j \in \{1, \dots, n\}$, then the j th entries of the vectors on the left hand sides and right hand sides of Equation (15.9) will be two distinct degree-2 univariate polynomials in r , and hence will disagree at a randomly chosen input with probability $1 - 2/|\mathbb{F}|$. Here, it is essential that the prover is forced to commit to the cross-term vector T before learning the verifier’s choice of $r \in \mathbb{F}$. Similarly, if either Equation (15.3) or (15.4) does not hold, then there is no vector T that the prover can commit to that would render every entry of the right hand and left hand sides of Equation (15.9) to be the same polynomials in r .

Formally, to be useful in designing a SNARK for iterative computation, we need to show that the above folding scheme is a proof of knowledge, meaning given any efficient prover that can convince the verifier of the validity of the folded instance with non-negligible probability, we can extract openings of the vectors w_1, E_1, w_2, E_2 that respectively satisfy the instances that were folded together (Equations (15.3) and (15.4)). We omit the details, as the paragraph above conveys the key intuition as to why a prover that does not behave as prescribed will, with overwhelming probability over the choice of r , be left to establish a false claim after the folding, namely that it can open the commitments to w and E to vectors satisfying Equation (15.9).¹⁷⁰

While this folding scheme is interactive, it is public coin, and hence can be rendered non-interactive via the Fiat-Shamir transformation (i.e, replace the verifier’s challenge with a hash of the public inputs and the prover’s message in the folding scheme).

15.5.3 A Large Non-Interactive Argument

A non-interactive argument of knowledge for an iterative computation $F^{(i)}(x)$ (with proof length linear in i) can be obtained by repeatedly applying the above folding scheme in the manner sketched in Section 15.5.1. This proof length is far too large to be interesting in applications, but it will be a useful object to have considered when we turn to designing the final SNARK (Section 15.5.4).

We will describe the proof as being produced and processed in “rounds”, even though it is non-interactive. Since there is no message sent from \mathcal{V} to \mathcal{P} , the entire proof is obtained by simply concatenating all prover messages across all “rounds”.

At the start of each “round” $j > 1$ of the protocol, there is already a “running folded instance” I of committed-RICS-with-slack-vector that captures the result of having folded across the first j rounds the RICS instances capturing the first $j - 1$ applications of F (as per Equation (15.2)), and the purpose of round $j > 1$ is to fold into this running instance the RICS capturing the j th application of F (again, as per Equation (15.2)). This means that at the start of round $j > 1$, the verifier will be tracking a commitment c_w to the “witness vector” w for I , and a commitment c_E for the “slack vector” E for I . The verifier at all times keeps track of the following variables:

- *round-count* (meant to track the number of applications j of F that have been processed so far).
- *prev-output* (meant to track $y_{j-1} = F^{(j-1)}(x)$)
- *cur-output* (meant to track $y_j = F^{(j)}(x)$)
- $u \in \mathbb{F}$ (meant to track the scalar u of the running folded instance I)

¹⁷⁰Readers are referred to the knowledge soundness analysis of the Bulletproofs polynomial commitment (Section 13.1.4) for an example of a knowledge soundness analysis for a folding scheme.

- $s \in \mathbb{F}^m$ (meant to track the public input s to the running folded instance I)
- c_w (meant to track the commitment to the witness vector w of the running folded instance I)
- c_E (meant to track the commitment to the slack vector E of the running folded instance I).

Let us introduce some notation to capture this state of affairs at the start of round j . We denote the prover's claim in running folded instance I at the very start of round j by

$$(A \cdot z) \cdot (B \cdot z) = u \cdot (C \cdot z) + E, \quad (15.15)$$

with the verifier's variables c_w, c_E being a commitments to w and E respectively, and recall that $z = (s, w)$. As per Equation (15.2), there is an R1CS instance that is satisfiable if and only if $F(y_{j-1}) = y_j$. This R1CS instance has the form

$$(A \cdot z_j) \cdot (B \cdot z_j) = C \cdot z_j, \quad (15.16)$$

where $z_j = (s_j, w_j) \in \mathbb{F}^m \times \mathbb{F}^{n-m}$, and $s_j = (y_{j-1}, y_j)$. Let us refer to this R1CS instance as I_j .

The prover's work in round j . At the start of round j , the prover sends the claimed value of y_j . This reveals to the verifier the public vector $s_j = (y_{j-1}, y_j)$, as \mathcal{V} learned the claimed value of y_{j-1} in the previous round. The prover also sends a commitment c_{w_j} to vector w_j . Together, these quantities specify the committed-R1CS instance I_j given in Equation (15.16). The purpose of round $j > 1$ is then to fold I_j into the running folded instance I . Accordingly, the prover sends a commitment c_T to the claimed cross-term T (Equation (15.11)). (In round $j = 1$, there is no folding operation to perform, as the verifier will simply set the running folded instance to I_1 ; see next paragraph for details).

How the verifier \mathcal{V} processes round j . Upon reading the prover's message in round $j = 1$, \mathcal{V} sets its variables in accordance with the running folded instance becoming I_1 . Specifically, \mathcal{V} sets *round-count* to 1, *prev-output* to x , *cur-output* to the claimed value of y_1 , u to 1, s to $(\text{prev-output}, \text{cur-output})$, c_w to c_{w_1} , and c_E to a commitment to the all-0s vector.

Upon receiving the prover's message in round $j > 1$, the verifier increments *round-count* from $j - 1$ to j , sets *prev-output* to *cur-output*, and updates *cur-output* to (the claimed value of) y_j . In a truly interactive protocol, the verifier would randomly choose the field element $r \in \mathbb{F}$ used for that round's folding operation and send it to the prover, but in the non-interactive setting, both prover and verifier can determine r via the Fiat-Shamir transformation as per Section 15.5.2. After r is chosen, using homomorphism of the vector commitment scheme, \mathcal{V} updates c_w to a commitment to $w + rw_j$ (as per Equation (15.6)).¹⁷¹ \mathcal{V} also updates c_E to a commitment to $E + rT$ (as per Equation (15.10)).¹⁷² \mathcal{V} updates $u \leftarrow u + r$ (as per Equation (15.7))¹⁷³ and updates $s \leftarrow s + r \cdot s_j$, where $s_j = (\text{prev-output}, \text{cur-output})$ (as per Equation (15.5)).

In this manner, after processing all i "rounds" of the proof, the verifier has computed a single folded committed-R1CS-with-slack-vector instance as per Equation (15.15), whose validity (up to a negligible soundness error) is equivalent to the validity of all i applications of F . In this final "round", the prover

¹⁷¹i.e., $c_w \leftarrow c_w \cdot (c_{w_j})^r$ where \cdot denotes the group operation of the multiplicative group over which the Pedersen vector commitments used by the protocol are defined (see Section 13.1.2).

¹⁷²Note that Equation (15.10) simplifies due to the fact that there is no slack vector in the R1CS instance of Equation (15.16)—equivalently, the slack vector is zero.

¹⁷³Note that Equation (15.7) simplifies due to the fact that in the right hand side of Equation (15.16), $C \cdot z_j$ is multiplied by the trivial scalar 1.

can establish the validity of the instance using any SNARK for committed-RICS-with-slack-vector. Such a SNARK can in turn be easily obtained from any SNARK for RICS satisfiability that commits to witness vectors via the same homomorphic vector commitment scheme used throughout the folding protocol. This includes the SNARK for RICS from Section 7.5 when combined with, say, the Bulletproofs polynomial commitment scheme (Section 13.1.4).

15.5.4 The Final SNARK: Nova

Unfortunately, the argument of the previous section yields a proof π that grows linearly with i , the number of applications of F . Roughly speaking, we now address this by forcing the SNARK prover to, in its own head, perform the verifier’s processing of π across i “rounds” of the protocol, and thereby avoid having the prover explicitly send π to the verifier.

Conceptual overview: folding as deferral of proof checking. The protocol of the previous section can be thought of as an argument system for incremental computation that works by reducing the checking of *all* applications of F (or more precisely, of RICS instances equivalent to F) to checking a single derived folding of the applications of F . That is, the validity of the single folded instance is equivalent to the validity of every one of the applications of F that the prover claims to have faithfully executed.

With this in mind, the (validity of) the running folded committed-RICS-with-slack-vector instance I at the start of each “round” $j > 1$ itself acts a “proof” π_j that $F^{(j-1)}(x) = y_{j-1}$. The folding procedure that occurs in “round” $j > 1$ should then be thought of as a way to *defer* checking the validity of π_j to a later point. Moreover, the folding has the effect of “accumulating” all i such checks into a single statement that can be checked at the same cost as performing any one of the validations individually. Specifically, the checks are deferred until all i foldings have occurred, at which point the prover finally establishes that the final running folded instance is valid.

The above method of “deferring/accumulating” the checking of each “proof” π_j is in contrast to the recursive-composition-of-SNARKs approach covered in Section 15.4, in which the prover explicitly proves that it verified a SNARK proof π_j in its own head for all $j = 1, \dots, i - 1$.¹⁷⁴ Intuitively, it is cheaper to defer/accumulate the checks than it is to actually explicitly perform each check, thereby reducing the overhead of recursion relative to the recursive-composition-of-SNARKs approach of Section 15.4 (we discuss exactly what is the overhead of recursion of Nova later).¹⁷⁵

The augmented function F' . Now we come to obtaining a SNARK from the folding scheme via recursive proof composition. Let us “augment” the computation of F to a larger computation F' that not only 1) applies F but also 2) does the verifier’s work in one step of the folding scheme. This is analogous to how, in Section 15.4, the honest prover in round j of proof generation applied a base SNARK \mathcal{O} to a circuit \mathcal{C}' that not only applied F to y_{j-1} , but also applied a verification circuit to the proof π_{j-1} computed in the previous round $j - 1$.

In more detail, F' will take as public input values for the variables maintained by the verifier in round j of the folding scheme (see the bulleted list in Section 15.5.3), and will also take as non-deterministic input the

¹⁷⁴More precisely, the prover establishes that it knows a π_j that would have convinced the SNARK verifier to accept. But this effectively means that the prover has itself applied the SNARK verifier’s accept/reject computation to π_j , given that the prover knows that the outcome of this computation is “accept”.

¹⁷⁵The deferral/accumulation of these checks is also analogous to earlier results such as Halo [BGH19], that deferred/accumulated only *part* of the verification of the SNARK proof π_j (namely the verification of evaluations of committed polynomials) via a folding-like procedure.

prover’s message in the folding scheme (except for the claimed value of y_j). F' will output the new values of the verifier’s variables in the folding scheme upon processing the prover’s message (see the paragraph entitled “How the verifier processes round j ” of Section 15.5.3). The one exception is that whereas the verifier in the folding scheme updates the value of the variable *cur-output* to a *claimed* value for y_j provided by the prover, F' will instead output the *actual* value of y_j . That is, F' will apply F to the relevant input and include the result in its output.

The SNARK. The final SNARK applies the folding-based proof of the previous section with F' in place of F .¹⁷⁶ But rather than outputting the entire proof, which consists of i “rounds”, the final SNARK proof provides only the information sent by the prover in the final “round”. This information comprises the following:

- A specification of the running folded instance I at the start of round i (Equation (15.15)), and a description of the final R1CS instance I_j to be folded in (Equation (15.16) with $j = i$). This latter description includes the claimed output of $(F')^{(i)}(x)$. This includes both the variable *round-count* (Section 15.5.3) and the claimed output y_i of $F^{(i)}(x)$. The SNARK verifier must confirm that *round-count* = i and reject if not, as this ensures that the proof actually refers to $F^{(i)}$ and not $F^{(j)}$ for some $j \neq i$. If all of the SNARK verifier’s remaining checks (described below) pass, then the verifier is convinced that indeed $y_i = F^{(i)}(x)$.
- The information provided by the prover in the “final round” of the protocol of the previous section to perform the final folding operation, specifically a commitment c_T to the cross-term used in this folding operation.
- A SNARK proof that the final folded instance is satisfiable.

In summary, the honest prover performs each “round” of the previous section’s protocol in its own head, only outputting a transcript of the final “round” of the protocol. This is analogous to how the prover in the recursive-SNARK solution of Section 15.4 for $F^{(i)}$ generated in its own head a sequence of SNARK proofs π_1, \dots, π_i , with each π_j attesting to a correct execution of F to input y_{j-1} (as well as knowledge of π_{j-1}). But ultimately, only the final proof π_i needs to be sent to the verifier to guarantee the correctness of the claimed output of $F^{(i)}$.

Essentially, each time that the Nova prover \mathcal{P} performs a folding operation in its own head, thereby folding I_j into the running folded instance I , the very next application of F' performs the verifier’s work in the folding operation (in addition to applying F for a $(j + 1)$ ’st time). This is the sense in which the final

¹⁷⁶This description elides the following subtlety, which requires a tweak to the definition of F' to address. The folding-scheme is applied to force the prover to faithfully compute $(F')^{(i)}$, which means that for $j \leq i$, the output of the $(j - 1)$ ’st application of F' has to be fed as public input to the j ’th application of F' . One “piece” of the output of F' is the folding-verifier’s variable s representing a “running folding” of all public inputs to previous applications of F' . This means that the vector s that is (just one piece of the) input to the j ’th application of F' has to be at least as big as the entire public input to the previous application of F' . But since there are other outputs of the $(j - 1)$ ’st application of F' as well (see the bulleted list of verifier values in Section 15.5.3), this forces the length of the public input to the j ’th application of F' to be *strictly bigger* than that of the previous application. Thus, the public input length for F' grows with each application of F' . To address this issue, one can modify F' to not include in its output $s \in \mathbb{F}^m$, but only a cryptographic hash $H(s)$, thereby ensuring that the output length of F' is independent of the length of s . F' will then take s as an additional non-deterministic input (rather than as public input) and as part of its computation it will confirm that s is indeed the pre-image of the associate public input value $H(s)$. In summary, without this modification, the public input size to F' grows iteration-by-iteration, because the vector s (the folding of prior public inputs) grows with each iteration. The modification replaces s in the input and output of F' with a hash $H(s)$, which addresses the issue because the size of the hash $H(s)$ does not depend on the length of the vector s .

Nova SNARK forces the prover of the previous section’s protocol to perform in its own head the verifier’s work of that protocol.

The overhead of recursion. In this SNARK, the overhead of recursion refers to the amount of extra work that F' does beyond simply applying F (or more precisely, the number of constraints in the RICS instance over field \mathbb{F}_p representing F' relative to the RICS instance over \mathbb{F}_p representing F). This extra work done in F' simply implements the verifier’s variable updates in the folding scheme (see the final paragraph of Section 15.5.3). This consists of a handful of field multiplications and additions over \mathbb{F}_p , one invocation of a cryptographic hash function per the Fiat-Shamir transformation, and the homomorphic updating of the two commitments c_w and c_E to obtain commitments to $w + rw_j$ (as per Equation (15.6)), and $E + rT$.

If a SNARK-friendly hash function is used for Fiat-Shamir, then it is the two homomorphic commitment updates that dominate the overhead of recursion. If the commitments are Pedersen vector commitments over a (multiplicative) group \mathbb{G} , then each of these updates requires one group exponentiation and one group multiplication; it is the two group exponentiations that dominate the cost (as a group exponentiation takes approximately $\log |\mathbb{G}| \approx \lambda$ group multiplications). This overhead of recursion is concretely cheaper than that of recursive-SNARK solutions considered earlier in this chapter (see Footnote 164 in Section 15.4).¹⁷⁷

Overall prover runtime. Assuming the number of iterations i is not very small, the prover’s runtime is dominated by the cost of computing a Pedersen vector commitment at every iteration $j \leq i$ to the witness vector w_j and the cross-term T . Both of these vectors have length at most n' , where n' is the number of rows of the RICS instance capturing F' (and as per the above overhead-of-recursion analysis, n' is quite close to n , the number of rows of the RICS capturing F alone). Hence, this is two multi-exponentiations per iteration. One does need to use a cycle of elliptic curves, but the curves need not be pairing friendly, ensuring fast group operations (see Section 15.2).

¹⁷⁷This description elides an important implementation issue that is essentially identical to the one described in Footnote 160 in the context of IVC from recursive SNARKs. Specifically, Pedersen vector commitments that are homomorphic over field \mathbb{F}_p are elements of an elliptic curve group \mathbb{G} in which the *scalar field* is \mathbb{F}_p and the *base field* is another field, \mathbb{F} . And group operations over \mathbb{G} can be efficiently implemented by a circuit or RICS defined over the base field, but unfortunately not the scalar field. Similar to Footnote 160, one way to sidestep this issue is to identify a cycle of curves \mathbb{G} and \mathbb{G}' with scalar and base fields \mathbb{F}_p and \mathbb{F} such that F is efficiently computable by circuits or RICS over *both* \mathbb{F}_p and \mathbb{F} . One then maintains two different sequences of RICS instances, with one sequence defined over field \mathbb{F}_p and the other defined over field \mathbb{F} . Since F is efficiently computable in RICS over both fields, one can efficiently define two different augmented functions, say, F' , and F'' , computing F and performing folding operations when commitments are sent over \mathbb{G}' and \mathbb{G} respectively. One then alternates performing folding operations on each sequence. Specifically, a folding of two committed-RICS-with-slack-vector instances defined over \mathbb{F}_p (and associated application of F') can be efficiently computed by F'' and hence by the RICS sequence defined over field \mathbb{F} , and similarly a folding operation of two instances defined over \mathbb{F} can be efficiently computed by F' and hence by the RICS sequence defined over \mathbb{F}_p . The final SNARK proof consists of the final folding operation for *both* sequences, and SNARK proofs for both sequences that the final folded instance is satisfied. Also similar to Footnote 160, if F is only efficiently implementable over \mathbb{F}_p one will still have two functions F' and F'' , but only F' will both apply F and implement folding; F'' will only implement folding. This will double the number of folding operations required to obtain a SNARK for $F^{(i)}$. Effectively only applications of F' perform the “useful work” of applying F ; applications of F'' are only used to “switch” which of the two fields folding operations can be efficiently computed over.

Chapter 16

Bird’s Eye View of Practical Arguments

In this survey, we have covered five approaches to the construction of practical general-purpose non-interactive argument systems. In each of the five, an underlying information-theoretically secure protocol is combined with cryptography to yield an argument. The first approach is based on the interactive proof for arithmetic circuit evaluation of Section 4.6 (the GKR protocol), the second is based on the MIPs for circuit or R1CS satisfiability of Sections 7.2 and 7.5, the third is based on the IOP for circuit or R1CS satisfiability of Section 9.3, and the fourth is based on the linear PCP of Section 14.4, transformed into a zk-SNARK via Section 14.5. The fifth is based on commit-and-prove techniques (Section 12.1), which can be viewed as combining a trivial static (i.e., **NP**) proof system with cryptographic commitments.

For each of the first three approaches (IP-based, MIP-based, and IOP-based), one can combine the information-theoretically secure protocol with any extractable polynomial commitment scheme of the protocol designer’s choosing to obtain a succinct argument (there is essentially just one technique to turn linear PCPs into publicly-verifiable SNARKs, based on pairings, see Section 14.5). For the IP-based and MIP-based argument systems, the polynomial commitment scheme must allow committing to *multilinear* polynomials. For the IOP-based argument system, the polynomial commitment scheme must allow committing to *univariate* polynomials. Of course, the resulting argument system will inherit the cryptographic and setup assumptions as well as any efficiency bottlenecks of the chosen polynomial commitment scheme.

We have covered three approaches to polynomial commitment schemes in this survey. The first is via IOPs for Reed-Solomon testing combined with Merkle hashing (Section 9.4), the second is based on Σ -protocols that assume hardness of the discrete logarithm problem (Section 13.1) and the third is based on the approach of KZG [KZG10] and uses pairings and a trusted setup (Section 13.2.2). Below, we simply call these respective approaches to polynomial commitments “FRI-based” (since FRI is the most asymptotically efficient known IOP for Reed-Solomon testing in terms of proof length)¹⁷⁸, “discrete-log-based”, and “KZG-based”.

16.1 A Taxonomy of SNARKs

The research literature on practical succinct arguments is a veritable zoo of built systems and theoretical protocols. In this section, we attempt to tame this zoo with a coherent taxonomy of the primary approaches

¹⁷⁸We covered an additional IOP-derived polynomial commitment scheme in Section 9.6, implicit in a system called Ligerio [AHIV17]. As discussed in Section 9.6.3, this scheme has significantly higher asymptotic verifier time and communication than FRI, though it is simpler and has a somewhat different cost profile for the prover. We do not discuss this IOP-derived polynomial commitment scheme further in this chapter.

that have been pursued.

In total, we have discussed as many as 11 broad approaches to designing general-purpose SNARKs, nine coming from each combination of {IP, MIP, IOP} and each of the three approaches to polynomial commitment schemes, the tenth coming from linear PCPs, and the eleventh from commit-and-prove techniques. A subset of research papers devoted to these approaches is as follows:

- (a) IPs (Section 4.6) combined with FRI-based (multilinear) polynomial commitments (Section 9.5) were explored in [ZXZS19], producing a system called Virgo.
- (b) IPs combined with discrete-log-based (multilinear) polynomial commitments (Section 13.1.3) were explored in [WTS⁺18], producing a system called Hyrax.
- (c) IPs combined with KZG-based (multilinear) polynomial commitments were explored in [ZGK⁺17a, ZGK⁺17b, XZZ⁺19] (Section 13.2.3), producing systems called zk-vSQL and Libra.
- (d) MIPs combined with all three approaches to (multilinear) polynomial commitments were explored in [Set19, SL20] (Section 7.5.2), producing systems called Spartan and Quarks that build on ideas from an MIP called Clover [BTVW14] (Section 7.2). These works have mostly focused on discrete-log-based polynomial commitments, because this yields the fastest prover and other efficiency benefits (see Section 13.4).

All of the above IPs and MIPs use or build on refinements of the GKR protocol developed over a series of works [CMT12, Tha13, BTVW14, Tha15, WJB⁺17, ZGK⁺18].

- (e) IOPs combined with FRI-based (univariate) polynomial commitments (Section 9.4) were explored in a lengthy series of works, most recently [BSCR⁺19, COS20, KPV19], producing systems called Aurora, Fractal, and Redshift (Section 9.3.2). Other related works in this series include [BSCGT13b, BSBHR18, SGKS20, BSBC⁺17, BBHR19, BCI⁺20].
- (f) IOPs combined with KZG-based (univariate) polynomial commitments (Section 13.2.2) were explored most recently in [CHM⁺19], producing a system called Marlin. A variety of other systems, including PLONK and Sonic [GWC19, MBKM19], have similar cost profiles to Marlin, in that they apply KZG-based commitments to a constant number of univariate polynomials, thereby generating proofs consisting of a constant number of group elements.
- (g) A very large number of systems have been derived from the linear PCP of Genarro, Gentry, Parno, and Raykova [GGPR13] (Section 14.5). These include [BCG⁺13, PHGR13]. The most popular variant of the SNARK derived from GGPR's linear PCP is due to Groth [Gro16], who obtained a proof length of just 3 elements of a pairing-friendly group, and proved the SNARK secure in the Generic Group Model that was briefly discussed in Section 13.2.2 ([FKL18] extended the security proof to the Algebraic Group Model).
- (h) Commit-and-prove based arguments have been studied in several works, e.g., [DIO20, BMRS20a, WYKW20]. These arguments are not succinct, and recent works on this approach yield interactive protocols; for both of these reasons, these arguments are not SNARKs.

There are a handful of approaches to SNARK design that do not necessarily fall into the categories above. One example is an approach called MPC-in-the-head, which takes protocols for secure multiparty computa-

tion (MPC) and transforms them into zero-knowledge arguments [IKOS09, AHIV17, GMO16].^{179,180} These protocols have a cost profile loosely analogous to commit-and-prove arguments (Approach (h)): they tend to have much larger proof sizes and higher verifier costs than the approaches above, but they can have good concrete costs on small input sizes, and good prover runtimes (competitive with or faster than other approaches). This had led, for example, to an interesting family of candidate post-quantum secure digital signatures, called Picnic [CDG⁺17, KZ20, DKP⁺19, KKW18, KRR⁺20]. Another example is that linear PCPs can be combined with *non*-pairing-based cryptosystems to yield designated-verifier (i.e., non-publicly-verifiable) SNARKs, including some based on the assumed hardness of lattice problems that are plausibly post-quantum secure [BCI⁺13].

In addition, designing more efficient polynomial commitment schemes is a central research goal, and any new extractable polynomial commitment scheme can be combined with the known IPs, MIPs, and IOPs that we cover to yield yet new SNARKs. Section 13.5 discusses several recently proposed polynomial commitment schemes that we do not discuss further in this survey.

16.2 Pros and Cons of the Approaches

The previous section contained bullet points that summarized 10 or 11 different approaches to the design of general-purpose succinct arguments. In this section, we aim to provide a concise overview of the pros and cons of these approaches.

Approaches minimizing proof size. There are two approaches that achieve proofs consisting of a constant number of group elements, captured in items (f) and (g) of the previous section—namely, IOPs combined with KZG-based polynomial commitments, and linear PCPs (transformed into SNARKs using pairing-based cryptography). The linear PCP approach is the ultimate winner in proof size, as its proofs consist of as few as 3 group elements [Gro16]. For comparison, Marlin [CHM⁺19] (which uses the former approach), produces proofs that are 4-6 times larger than that of Groth’s SNARK [Gro16].

The downsides of the two approaches are also related. First, both require a trusted setup (as they make use of a structured reference string), which produces toxic waste (also called trapdoors) that must be discarded to prevent forgeability of proofs. In the case of IOPs combined with KZG-based polynomial commitments, the downsides of the SRS are not as severe as for linear PCPs, for two reasons. First, the SRS for the former approach is *universal*: a single SRS can be used for *any* R1CS-satisfiability or circuit-satisfiability instance up to some specified size bound. This is because the SRS simply consists of encodings of powers of a random field element τ , and hence is independent of the circuit or R1CS instance. In contrast, the SRS in the linear PCP approach is *computation-specific*: in addition to including encodings of powers-of- τ , the SRS in the linear PCP approach also has to include encodings of evaluations of univariate polynomials capturing the wiring pattern of the circuit or the matrix entries specifying the R1CS instance.¹⁸¹ Second, the SRS for the former approach is *updatable* (see Section 13.4 for discussion of this notion), while the SRS for

¹⁷⁹MPC-in-the-head actually yields IOPs [AHIV17]. Hence, it need not be viewed as distinct from IOP-based approaches. In fact, Ligerio [AHIV17] is derived from the MPC-in-the-head approach, and we covered the central component implicit in Ligerio, namely an IOP-based polynomial commitment scheme that is different from FRI, in Section 9.6.

¹⁸⁰Other related techniques also derive zero-knowledge proofs from MPC protocols [FNO15, HK20, JKO13], with broadly similar cost profiles to MPC-in-the-head (long proofs and verification time, but good prover runtime and small hidden constants).

¹⁸¹One can render universal a SNARK with a computation-specific SRS by applying the SNARK to a so-called universal circuit, which takes as input both a *description* of another circuit \mathcal{C} and an input-witness pair (x, w) for \mathcal{C} and evaluates \mathcal{C} on (x, w) . This introduces significant overhead, despite several mitigation efforts [BCTV14b, KPPS20]; see [WSR⁺15] for some concrete measurements of overhead.

the linear PCP approach is not, again owing to the fact that the SRS contains elements other than encodings of powers-of- τ .

The second downside of both of these two approaches is that they are computationally expensive for the prover, for two reasons. First, in both approaches, the prover needs to perform FFTs or polynomial division over vectors or polynomials of size proportional to the circuit size S , or number K of nonzero entries of the matrices specifying the R1CS instance. This is time-intensive as well as highly space intensive and difficult to parallelize and distribute [WZC⁺18]. Second, in both approaches the prover also needs to perform $\Theta(S)$ or $\Theta(K)$ many group exponentiations in a pairing-friendly group. See Section 13.4 for discussion of the concrete costs of these operations.

In contrast to the two approaches that minimize proof size, none of the remaining approaches require trusted setup unless they choose to use KZG-based polynomial commitments. That is, they use a uniform reference string (URS) rather than a structured reference string. SNARKs that use a URS rather than an SRS are also called *transparent*.

Approaches that are plausibly quantum secure. The next natural category of approaches to consider are those that are plausibly post-quantum secure. Amongst the approaches delineated in the previous section, these comprise those involving the FRI-based approach to polynomial commitments (i.e., IPs, MIPs, and IOPs combined with FRI-based polynomial commitments, or more generally any polynomial commitment that is binding against quantum adversaries), as well as commit-and-prove based arguments if the commitment scheme used is binding against quantum adversaries.¹⁸² For the latter category, the main additional benefit is a prover that is somewhat faster than approaches yielding succinct arguments, but the arguments are not succinct (large proofs), and many systems based on this approach [DIO20, BMRS20a, WYKW20] only yield interactive (and hence also non-publicly-verifiable) arguments.

Within the former category, beyond post-quantum security, the pros and cons of these approaches match those of the FRI-based polynomial commitment scheme (Section 13.4). Specifically, the main additional advantage is that it uses a short URS (i.e., the public parameters simply specify a constant number of cryptographic hash functions). The downsides are computational expense and proof size inherited from the polynomial commitment scheme. When combining MIPs and IOPs with FRI-derived polynomial commitments, the prover has to commit to a polynomial of size $\Theta(S)$ or $\Theta(K)$, which requires performing FFTs over vectors of length $\Theta(S)$ and $\Theta(K)$, and this is a key bottleneck for reasons described above. When combining IPs with the FRI-derived polynomial commitments, the FFTs are applied only to the witness and not the entire circuit or R1CS-system. This is still expensive unless the witness is small. These approaches also lead to proofs that are much larger than the first category of approaches discussed above. For example, FRI alone (in fact, even *one* invocation of the query phase of FRI) requires the prover to send logarithmically many Merkle-tree authentication paths, each of which consists of logarithmically many cryptographic hash evaluations; moreover, the query phase must be repeated dozens or hundreds of times to ensure low soundness error (see Section 9.4).

Remaining approaches. The remaining approaches in our taxonomy are to combine IPs or MIPs with either discrete-logarithm-based polynomial commitments or KZG-based polynomial commitments. A major advantage of these approaches is that the IPs and MIPs covered in this survey (all of which are refinements of, or inspired by, the GKR protocol) are currently the only known information-theoretically secure protocols for circuit- or R1CS-satisfiability in which the prover performs a number of field operations that is *linear*

¹⁸²The MPC-in-the-head approach, by virtue of yielding IOPs (see Footnote 179), also gives plausibly post-quantum secure protocols [GMO16, AHIV17].

in the size S of the circuit or number K of nonzero matrix entries specifying the R1CS. The linear PCPs and IOPs covered in this survey require the prover to perform FFTs or polynomial division on vectors or polynomials of size S or K , and these require $\Theta(S \log S)$ and $\Theta(K \log K)$ field operations, and are concretely a major bottleneck in prover scalability.¹⁸³

When combining the IPs and MIPs with discrete-logarithm based polynomial commitments, one obtains *succinct* arguments with the fastest known prover, at least concretely for large enough statement sizes. In particular, by combining the MIP of Section 7.5.2 with the polynomial commitments described in Section 13.1, one obtains a prover for R1CS-SAT on K -sparse matrices that performs $O(K)$ field operations and group multiplications and a group exponentiation of size $O(K)$. If the polynomial commitment scheme used is Dory (Section 13.1.5), the verifier runs in polylogarithmic time, after a transparent pre-processing phase to generate a structured verification key, and the proof length is $O(\log K)$ group elements. If another discrete-logarithm based polynomial commitment is used (e.g., the ones of Section 13.1.3 or 13.1.4), the verification time is slower, but there is no pre-processing phase to generate structured parameters, and the cryptographic group used need not be pairing-friendly, which can lead to improved concrete efficiency for the prover. If using KZG-based rather than discrete-logarithm-based polynomial commitments, the verification time is comparable to that obtained by using Dory, but a trusted setup is required.

The comparison between the resulting IP-based and MIP-based arguments is the following. The downside of the IP approach (based on refinements of the GKR protocol, see Section 4.6) is that it applies only to *layered arithmetic circuits*, and the proof length and verifier time grows linearly with the depth of the circuit (i.e., proof length is $O(d \log S)$).¹⁸⁴ Hence, it only yields short proofs for small-depth circuits. In contrast, the MIP approach (Sections 7.2 and 7.5) achieves a proof length and verifier time of $O(\log S)$, and extends to R1CS-satisfiability, not just (small-depth) arithmetic circuit satisfiability. The benefit of the IP approach over the MIP approach is that, when considering a circuit-satisfiability instance of the form $\mathcal{C}(x, w) = y$, the polynomial commitment scheme only needs to be applied to the multilinear extension of the witness w , not to the multilinear extension of a valid circuit transcript (which has size $S = |\mathcal{C}|$). This means that for circuits where $|w| \ll S$ (i.e., the witness is much smaller than the full circuit \mathcal{C}), IPs can be combined with any of the polynomial commitment schemes discussed in this survey to yield an argument system prover that performs $O(S)$ many field operations, and $O(|w|) = o(S)$ many cryptographic hash evaluations or group operations.

In summary, MIP-based protocols such as Spartan [Set19] are generally preferable over IP-based protocols such as Hyrax [WTS⁺18], Libra [XZZ⁺19], or Virgo [ZXZS19] for deep circuits or circuits with large witnesses, or in settings where R1CS-satisfiability is a more convenient intermediate representation than arithmetic circuits. IP-based protocols are typically preferable for applications that lend themselves to small-depth circuits with short witnesses.

On pre-processing and work-saving for the verifier. The approaches requiring a trusted setup (i.e., linear PCPs, or combining any IP, MIP, or IOP with KZG-based polynomial commitments) inherently require a pre-processing phase to generate the SRS, and this takes time proportional to the size of the circuit or R1CS

¹⁸³Very recent work of Kothapalli, Masserova, and Parno [KMP20] shows how to combine the IOPs of Section 9.3.2 with any polynomial commitment scheme in a manner that avoids the use of FFTs by the prover (so long as the polynomial commitment scheme itself avoids FFTs). However, their solution requires the prover to perform a multiexponentiation of size $\Theta(K)$ in the field. Using Pippenger’s multiexponentiation algorithm, this equates to $\Theta(K \log(|\mathbb{F}|) / \log K)$ field operations. If $|\mathbb{F}|$ is superpolynomial in K (i.e., $K^{\omega(1)}$), as is necessary to ensure negligible soundness error, this is super-linearly many field operations.

¹⁸⁴Very recent work mitigates the need for layered circuits, though at the cost of an increase in verification time or proof length [ZWZZ20].

instance and must be performed by a trusted party.¹⁸⁵ But the other approaches (combining any IP, MIP, or IOP with FRI-based or discrete-log-based polynomial commitments) can achieve a work-saving verifier *without pre-processing*, if applied to computations with a “regular” structure. By work-saving verifier, we mean that \mathcal{V} runs faster than the time required simply to check a witness—in particular, \mathcal{V} ’s runtime is sublinear in the size of the circuit or R1CS instance under consideration. For example, the MIP of Section 7.2 achieves a work-saving verifier without pre-processing so long as the multilinear extensions add and mult of the circuit’s wiring predicate can be efficiently evaluated at any input, and any RAM of runtime T can be transformed into such a circuit of size $\tilde{O}(T)$ (Chapter 5).

That said, not all implementations of these approaches seek to avoid pre-processing for the verifier. One reason for this is that guaranteeing that the intermediate representation (whether a circuit, R1CS instance, or other representation) has a sufficiently regular structure to avoid pre-processing can introduce large concrete overheads to the representation size; we briefly discussed such overheads in Section 5.6.4. Another is that “paying for” an expensive pre-processing phase can enable improved efficiency in the online phase of the protocol. For example, a primary ethos of SNARKs derived from linear PCPs is that, while it is expensive to generate the (long) SRS and distribute it to all parties wishing to act as the prover, checking proofs is extremely fast. Specifically, proof checking consists of only a constant number of group operations and bilinear map (pairing) evaluations.

To give a few examples from the research literature, STARKs [BBHR19] implement an IOP specifically designed to avoid pre-processing and achieve a polylogarithmic time verifier for *any* computation, with considerable effort devoted to mitigating the resulting overheads in the size of the intermediate representation. Although STARKs achieve considerable improvements over earlier instantiations of this approach [BSBC⁺17], the resulting intermediate representations remain very large in general. Meanwhile, many IP and MIP implementations avoid or minimize pre-processing in data parallel settings [Tha13, WJB⁺17, WTS⁺18] (see Section 4.6.7). These systems are able to exploit data parallel structure to ensure that the verifier can efficiently compute the information it needs about the computation in order to check the proof. Specifically, the time required for the verifier is independent of the number of parallel instances executed. They achieve this without incurring large concrete overheads in the size of the intermediate representation (see Section 5.6.4 for a sketch of how such overheads can arise when supporting work-saving verifiers for arbitrary computations).

Still other systems, such as Marlin [CHM⁺19], RedShift [KPV19], and Spartan [Set19] implement IOPs and MIPs targeted for the pre-processing setting, where a trusted party can commit to polynomials encoding the wiring of the circuit or R1CS instance during pre-processing, and thereafter, every time the circuit or R1CS is evaluated on a new input, the verifier can run in time sublinear in the circuit size. Finally, many systems do not seek a work-saving verifier even after potential pre-processing—these include [BSCR⁺19, AHIV17, BBB⁺18, BCC⁺16, Gab19a].

16.3 Other Issues Affecting Concrete Efficiency

There are many subtle or complicated issues that can affect the concrete efficiency of a SNARK. This section provides an overview of some of them.

¹⁸⁵The one partial exception is that combining IPs with KZG-based polynomial commitments has a setup phase of cost proportional to the size of the witness w rather than the entire circuit \mathcal{C} .

16.3.1 Field choice

A subtle aspect of the various approaches to SNARK design that can have a significant effect on practical performance is the many ways in which the designer’s choice of field to work over can be limited. One reason this matters is that for certain fields, addition and multiplication are particularly efficient on modern computers. For example, when working over Mersenne-prime fields (\mathbb{F}_p where p is a prime of the form $2^k - 1$ for some positive integer k), reducing an integer modulo p can be implemented with simple bit-shifts and addition operations, and field multiplication can be implemented with a constant number of native (integer) multiplications and additions, followed by modular reduction. Mersenne primes include $2^{61} - 1$, $2^{127} - 1$, and $2^{521} - 1$. Similarly fast arithmetic can be implemented more generally using any pseudo-Mersenne prime, which are of the form $2^k - c$ for small odd constant c (e.g., $2^{224} - 2^{96} + 1$). In contrast, modular reduction in an arbitrary prime-order field potentially requires division by p , and this is typically slower than reduction modulo pseudo-Mersenne primes by a factor of at least 2.¹⁸⁶ As another example of fields with fast arithmetic, some modern CPUs have built-in instructions for arithmetic operations in fields of sizes including 2^{64} and 2^{192} .

Limitations on the choice of field size for SNARKs come in multiple ways. Here are the main examples.

Guaranteeing soundness. All of the IPs, IOPs, MIPs, and linear PCPs that we have covered have soundness error that is at least $1/|\mathbb{F}|$ (and often larger by significant factors). Of course, so long as the soundness error is at most, say, $1/2$, the soundness error can always be driven to $2^{-\lambda}$ by repeating the protocol λ times, but this is expensive (often, only certain “soundness bottleneck” components need to be repeated, and this can mitigate the blowup in some costs, see for example Section 9.4). Regardless, $|\mathbb{F}|$ must be chosen sufficiently large to ensure the desired level of soundness.

Limitations coming from polynomial commitments. SNARKs making use of discrete-logarithm-based or KZG-based polynomial commitments (Chapter 13), or linear PCPs (which are compiled into SNARKs via pairings) must use a field of size equal to the order of the cryptographic group that the polynomial commitment is defined over. In contrast, SNARKs using polynomial commitment schemes derived from Reed-Solomon testing do not suffer such limitations, as the only cryptographic primitive they make use of is a collision-resistant hash function (to build a Merkle-tree over the evaluations of the polynomial to be committed), and such hash functions can be applied to arbitrary data.

Limitations coming from FFTs. SNARKs derived from IOPs (Chapter 9) and linear PCPs (Chapter 14) require the prover to perform FFTs over large vectors, and different finite fields support FFT algorithms of different complexities. In particular, FFTs running in time $\tilde{O}(n)$ on vectors of length n for prime fields \mathbb{F}_p are only known if $p - 1$ has many small prime factors.

Many, but not all, desirable fields do support fast FFT algorithms¹⁸⁷. Indeed, many prime fields \mathbb{F}_p do have $p - 1$ divisible by small prime factors. As another example, all fields of characteristic 2 do have efficient FFT algorithms, though until relatively recently, the fastest known algorithm ran in time $O(n \log n \log \log n)$. The extra $\log \log n$ factor was removed only in 2014 by Lin et al. [LAHC16].

¹⁸⁶More information on efficient techniques for modular reduction in arbitrary prime-order fields can be found, for example, at https://en.wikipedia.org/wiki/Montgomery_modular_multiplication.

¹⁸⁷By desirable, we either mean that the field supports fast arithmetic and meets the other desiderata described in this section, or that a particular cryptographic application calls for use of the field, say, because a SNARK is being used to prove a statement about an existing cryptosystem that performs arithmetic over the field.

A related issue is that IOP-derived SNARKs require the field to have multiplicative or additive subgroups of specified sizes (see for example the IOP for R1CS-satisfiability, which requires \mathbb{F} to have a subgroup H of size roughly the number variables of the R1CS system, and a second subgroup $L_0 \supset H$ of size a constant factor larger than H). Again, many desirable fields contain subgroups of appropriate sizes, so this is also not typically a major limitation. For example, a field of size 2^k has additive subgroups of size $2^{k'}$ for every $k' \leq k$ (see Remark 9.2 in Section 9.3.1).

Limitations coming from program-to-circuit transformations. IOP-derived SNARKs that seek to emulate arbitrary computer programs (Random Access Machines (RAMs)) while being work-saving for the verifier and avoiding pre-processing typically use transformations from RAMs to circuits or other intermediate representations that only work over fields of characteristic 2. We saw an example of this in Section 8.4.1, and modern instantiations such as STARKs [BSBC⁺17, BBHR19] also have this property.

Other considerations in field choice. There are other considerations when choosing a field to work over, beyond the limitations described above. For example, as discussed in Section 5.5.4, a prime field of size p naturally simulates integer addition and multiplication so long as one is guaranteed that the values arising in the computation always lie in the range $[-p/2, p/2]$ (if the values grow outside of this range, then the field, by reducing all values modulo p , will no longer simulate integer arithmetic). Such an efficient simulation is not possible in fields of characteristic 2. Conversely, addition in fields of characteristic 2 is equivalent to bitwise-XOR. Hence, aspects of the computation being fed through the proof machinery will affect which choice of field is most desirable: arithmetic-heavy computations may be more efficiently simulated when working over prime fields, and computations heavy on bitwise operations may be better suited to fields of characteristic 2.

Example field choices. To give some examples from the literature: Aurora [BSCR⁺19], which is based on IOPs, chooses to work over the field of size 2^{192} . This is large enough to provide good soundness error while supporting FFT algorithms requiring $O(n \log n)$ group operations, and some modern processors have built-in support for arithmetic over this field. Virgo [ZXZS19] chooses to work over the field of size p^2 where $p = 2^{61} - 1$ is a Mersenne prime, to take advantage of the fast field operations offered by such primes. STARKs [BBHR19] chooses to work of the field of size 2^{64} . This field is not large enough to ensure cryptographically-appropriate soundness error on its own, so aspects of the protocol are repeated several times to drive the soundness error lower.

The three systems above use FRI-based polynomial commitments, meaning they do not have to work over a field of size equal to the order of some cryptographically-secure group. SNARKs based on pairings or discrete-logarithm-based polynomial commitments are not able to work over these fields.

Hyrax [WTS⁺18] and Spartan [Set19], both of which combine IPs or MIPs with discrete-logarithm-based polynomial commitments, work over the field whose size is equal to the order of (a subgroup of) the elliptic curve group Curve25519 [Ber06] (see Section 11.1.2), with this group chosen for its fast group arithmetic and popularity.

Systems that use pairings (e.g., all linear-PCP-derived SNARKs, as well as zk-vSQL, vRAM, Libra, and PLONK, all of which use KZG-based polynomial commitments) work over a field of size equal to the order of (a subgroup of) chosen pairing-friendly elliptic curves. There have been significant efforts to design such pairing-friendly curves with fast group arithmetic while ensuring, e.g., that the order of the chosen subgroup is a prime p such that the field \mathbb{F}_p supports fast FFTs (see for example [BCG⁺13]).

The choice of field can make a significant concrete difference in the efficiency of field arithmetic. For example, experiments in [Set19, ZXZS19] suggests that the field used in Virgo (of size $(2^{61} - 1)^2 \approx 2^{122}$) has arithmetic that is at least 4x faster than the field used in Hyrax and Spartan (of size close to 2^{252}). Much of this 4x difference can be attributed to the fact that Virgo’s field is roughly square root of the size of Hyrax and Spartan’s, and hence field multiplications operate over smaller data types. However, some of the difference can be attributed to extra structure in the Mersenne prime $2^{61} - 1$ that is not present in the prime order field used by Hyrax and Spartan.

16.3.2 Relative Efficiency of Different Operations

Of course, the speed of field arithmetic is just one factor in determining overall runtime of a SNARK. In some SNARKs, the bottleneck for the prover is performing FFTs over the field, in others the bottleneck is group operations, and in still others the bottleneck may be processes that have nothing to do with the field choice (e.g., building a Merkle tree). To give one example, in SNARKs for R1CS-satisfiability derived from IOPs, the prover typically has to perform an FFT over a vector of length $\Theta(K)$, where K is the number of nonzero matrix entries of the R1CS system, and also must build one or more Merkle trees over vectors of length $\Theta(K)$. For large values of K , the $O(K \log K)$ runtime of the FFT will be larger than the time required to perform the $\Theta(K)$ evaluations of a cryptographic hash function that are needed to build the Merkle tree(s). But for small values of K , the $\log K$ factor in the FFT runtime may be concretely smaller than the time required to evaluate a cryptographic hash evaluation (particularly if the field supports fast arithmetic, ensuring the hidden constant in the FFT runtime is small). So which part of the protocol is the bottleneck (FFT vs. Merkle-tree building) likely depends on how large a computation is being processed.

As another example, the bottleneck for the prover in a SNARK for R1CS-satisfiability in a system such as Spartan [Set19] (which combines MIPs with discrete-logarithm-based or KZG-based polynomial commitments over group \mathbb{G}) is typically in performing one multi-exponentiation of size proportional to K .¹⁸⁸ Via Pippenger’s algorithm, the multiexponentiation can be done using $O(K \log(|\mathbb{G}|)/\log(K))$ group multiplications. In most other SNARKs (e.g., IOP-based or linear-PCP-based), the prover would have to at least perform an FFT over a vector of length at least K , and this will cost $O(K \log K)$ field operations.

For small R1CS instances, the FFT is likely to be faster than the multi-exponentiation, for three reasons. First, each group operation in a cryptographically secure group \mathbb{G} is often an order of magnitude more expensive than a field multiplication. Second, when K is small, $\log(|\mathbb{G}|)/\log(K) \gg \log K$, so even ignoring differences in the relative cost of a group vs. field operation, $O(K \log(|\mathbb{G}|)/\log K)$ is larger than $O(K \log K)$. Third, if the SNARK uses FRI-based polynomial commitments, it has the flexibility to work over a field whose size is not the order of an elliptic-curve group, and these fields can potentially support faster arithmetic. However, once K is large enough that $\log |\mathbb{G}| \ll \log^2 K$, the $O(K \log K)$ field operations required by the FFT will take more time than the $O(K \log(|\mathbb{G}|)/\log(K))$ group multiplications required to perform the multiexponentiation.

¹⁸⁸This is the case if using the polynomial commitment schemes of Section 13.1.2, 13.1.4, or 13.1.5; if using the related polynomial commitment scheme of Section 13.1.3, the prover instead does \sqrt{K} multiexponentiations each of size \sqrt{K} , which has the same asymptotic cost as a single multiexponentiation of length K .

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